Subquadratic algorithms for the general collision detection problem

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Abstract

We present the first subquadratic collision detection algorithm for simultaneously moving geometric objects which works in a fairly general setting. Geometric objects are regarded as rigid bodies in 3-space and are represented by unions of triangles (polyhedra) or unions of spheres (molecules). The motions of all objects are specified by polynomial functions which describe their position and orientation at any point in time. The general framework we develop for the solution of our specific problem is interesting of its own because it may be applicable for a wide range of other problems which require the solution of systems of polynomial (in)equalities.

Classification: algorithms and data structures, computational geometry

1 Introduction

Collision detection is prerequisite for simulating the physically correct behaviour of real world processes. It is an important tool in the field of "mechanical computer aided engineering" and in the field of "computational molecular biology". There it is essential to detect unintentional interferences between objects as early as possible. Moreover real time collision detection is still a major bottleneck in most virtual reality applications. That is the reason why efficient collision detection algorithms must be developed.

Let us consider the problem of collision detection abstractly.

> Given two simultaneously moving objects B_1 and B_2 with well defined geometric form and trajectory, decide whether their motion is collision-free.

To be more specific we assume that the objects B_i are rigid bodies represented by a set of triangles or by a set of spheres. The complexity of B_i is simply measured by the cardinality of the defining set. The trajectory of each B_i is specified in advance by a polynomial function $\mathcal{C}_i(t)$ which describes its configuration at time $t \in [0,1]$. A collision between two molecules/polyhedra occurs if two spheres/triangles from different objects collide. Thus the trivial algorithm to detect a collision simply compares every pair of spheres/triangles and runs in time $O(N^2)$. Here $N = |B_1| + |B_2|$ denotes the total complexity of both objects. The contribution of this paper consists in the proof that it is possible to decide in time $o(N^2)$ whether the objects collide. This result can be seen as a generalization of the results obtained in [6].

This paper is organized as follows. First we describe the mathematical model which underlies our approach. Next we summarize previous results for some special cases of the collision detection problem, which build the kernel of the general strategy. In particular we introduce the concept of linearization. In section 4 we show how to reduce the collision detection problem for two spheres or two triangles to the task to determine the number of real roots of a polynomial satisfying several polynomial inequalities.

In contrast to the solutions proposed in [1, 2], which are based on Sturm sequences, we apply some long known theorems of Jacobi, Hermite and Sylvester which deal with the existence of real roots of a system of univariate polynomial equations/inequalities (see sections 5, 6). It turns out, that this technique is especially suitable to find an appropriate linearization. The combination of the data structures in section 3 and this linearization yields a subquadratic algorithm.

2 Preliminaries

The configuration of a rigid body B_i in 3-space can be easily specified by the position and orientation of its local coordinate frame. Let vector $\mathbf{o}_i \in \mathbb{R}^3$ denote the position of B_i's reference point and quaternion $\mathfrak{r}_i \in \mathbb{R}^4$ its orientation. Quaternion calculus provides an elegant

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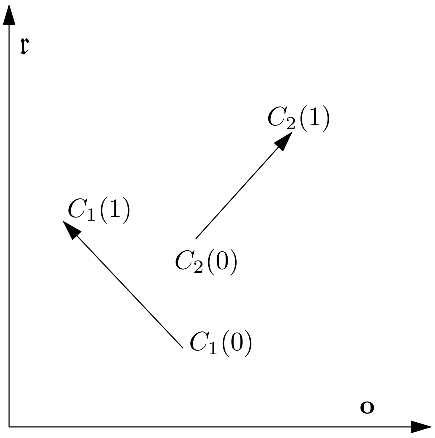

Figure 1: Configuration space

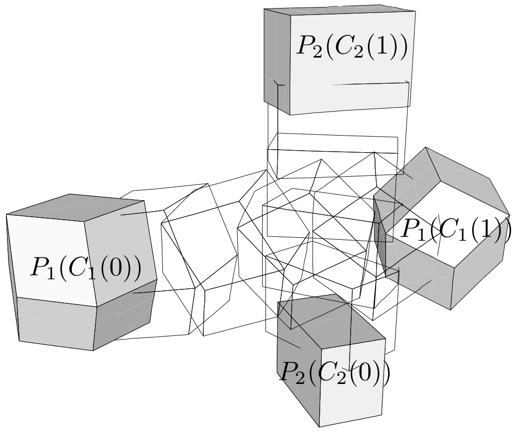

Figure 2: Physical space

way of algebraically specifying orientations in 3-space analogous to complex numbers in 2-space. (For a short review of quaternion calculus see appendix A.) Thus the configuration $\mathcal{C}_i(t)$ at time t can be represented as a tuple $(\mathbf{o}_i(t), \mathfrak{r}_i(t)) \in \mathbb{R}^3 \times \mathbb{R}^4$. For a moving body B_i the configuration $\mathcal{C}_i(t)$ varies with time and defines a unique trajectory. For simplicity we assume that the motion is confined to the time interval $[0, 1]$.

$$\begin{aligned} \mathcal{C}_i(t) : \quad & [0,1] \mapsto \mathbb{R}^3 \times \mathbb{R}^4 \\ & \mathcal{C}_i(t) = (\mathbf{o}_i(t), \mathfrak{r}_i(t)) \\ & \text{where } \mathbf{o}_i(t) \in \mathbb{R}[t]^3 \text{ and } \mathrm{r}_i(t) \in \mathbb{R}[t]^4 \end{aligned}$$

I.e. the components of $\mathbf{o}_i(t)$ and $\mathfrak{r}_i(t)$ are polynomials in t. Let d_i and d_i' denote their degree. In vector-matrix notation the trajectory of a single point $\mathbf{x} \in B_i$ is given by

$$\mathbf{x}(t) \;=\; \mathbf{R}_i(t)\mathbf{x} + \mathbf{o}_i(t). \tag{1}$$

where $\mathbf{R}_i(t) \in \mathbb{R}(t)^3 \times \mathbb{R}(t)^3$ is the rotation matrix corresponding to the quaternion $\mathfrak{r}_i(t)$ (see equation (3)).

The following table shows the simplest kinds of motion of a rigid body B_i for the case of constant or linearly changing positions and orientations and in figure 1 and 2 such motions are depicted in configuration and in physical space.

$d_i = 0$	$d_i' = 0$	B_i remains stationary
$d_i = 1$	$d_i' = 0$	B_i is translated in a fixed direction
$d_i = 0$	$d_i' = 1$	B_i rotates about a fixed axis
$d_i = 1$	$d_i' = 1$	B_i's motion is a superposition of a translation and a rotation

3 General approach

For the collision detection problem for a stationary and a translationally moving polyhedron respectively a polyhedron rotating about a fixed axis subquadratic algorithms were presented in [6]. In this paper we will use the same data structures for solving the general version of the problem. For completeness we give a short overview of the approach in [6]. The basis of our algorithm is an efficient solution of the following subproblem:

> Let $\mathcal{S}$ be a set of rigid bodies of the same type and let $\mathcal{Q}$ be a second set of rigid bodies of another type. Build a data structure that, given a query object $Q \in \mathcal{Q}$ decides quickly whether the object Q collides with an object from the set $\mathcal{S}$ during its motion. We call this the *on-line collision problem for $\mathcal{Q}$ with respect to $\mathcal{S}$.*

Our strategy is to reduce the collision problem to a problem for other objects that do not move and then solve the latter by known techniques. This is done by the concept of linearization. To find a *linearization* of the collision problem means to establish the equivalence

$$\begin{aligned} & [\exists t \in [0,1] :\; S(t) \cap Q(t) \neq \emptyset] \\ \iff & \bigvee_{i=1}^{dis} \bigwedge_{j=1}^{con} \left[\sum_{k=1}^{dim} \sigma_k^{ij}(S)\, \delta_k^{ij}(Q) \bowtie 0 \right], \end{aligned} \tag{2}$$

where $\bowtie \in \{<, >, =\}$, *dis*, *con*, *dim* are positive constants, and $\sigma_k^{ij}(S)$ respectively $\delta_k^{ij}(Q)$ are rational functions of constant degree depending on the kind of motion and kind of objects.

Having such a linearization we map the objects $S \in \mathcal{S}$ into the points $p^{ij} := (\delta_1^{ij}(S), \delta_2^{ij}(S), \ldots, \delta_{dim}^{ij}(S))$ in $\mathbb{R}^{dim}$ and the query object Q into the hyperplanes $h^{ij} := (\sigma_1^{ij}(Q), \sigma_2^{ij}(Q), \ldots, \sigma_{dim}^{ij}(Q))$ in the same space. Then we can think of any $\sum_{k=1}^{dim} \sigma_k^{ij}(Q)\, \delta_k^{ij}(S) \bowtie 0$ as the condition, that (depending on $\bowtie$) the point p^{ij} lies on the hyperplane h^{ij} respectively in a halfspace bounded by h^{ij}.

Therefore the linearization (2) leads to a combination of several halfspace range searching problems. A general notation for such combined search problems was first introduced in [4]:

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ be a set of N points in $\mathbb{R}^{dim}$, let $\mathcal{R}$ denote the set of all simplices in $\mathbb{R}^{dim}$, let $\mathcal{S} = \{s_1, \ldots, s_N\}$ be a set of N objects, and let $\mathcal{Q}$ denote a set of queries on $\mathcal{S}$. The *composed query problem* $(\mathcal{S}', \mathcal{Q}')$ is defined as follows: $\mathcal{S}' = \{(p_i, s_i); 1 \le i \le N\}$, $\mathcal{Q}' = \mathcal{R} \times \mathcal{Q}$ and the answer set for a query $(R, Q) \in \mathcal{Q}'$ is given by $\{(p, s);\ (p, s) \in \mathcal{S}'$ and $p \in R$ and $s \in Q\}$. We also say that $(\mathcal{S}', \mathcal{Q}')$ is obtained from $(\mathcal{S}, \mathcal{Q})$ by *simplex composition*.

Simplices in *dim*-space are the intersection of at most $dim + 1$ many halfspaces. Therefore we can w.l.o.g. consider simplex compositions where the simplices are halfspaces. In this case we also use the term *halfspace composition*.

Because each conjunction of (2) can be interpreted as the composition of *con* halfspace range searching problems we can find the objects in $\mathcal{S}$ satisfying a particular conjunction by applying halfspace composition *con* times. The disjunctions of (2) correspond to the union of ranges.

In his Ph.D. thesis [4] Marc van Kreveld investigated efficient solutions for simplex composition[1] of query problems:

Theorem 1 ([4]) *Let $\mathcal{P}$ be a set of N points in dim-space, and let $\mathcal{S}$ be a set of N objects in correspondence with $\mathcal{P}$. Let T be a data structure on $\mathcal{S}$ having building time $b(N)$, size $s(N)$ and query time $q(N)$. For an arbitrary small constant $\epsilon > 0$, the application of simplex composition on $\mathcal{P}$ to T results in a data structure D of building time $O(M^\epsilon(M + b(N)))$, size $O(M^\epsilon(M + s(N))$ and query time $O(N^\epsilon(q(N) + N/M^{1/dim}))$ for every fixed M such that $N \le M \le N^{dim}$, assuming that $s(N)/N$ is non-decreasing and $q(N)/N$ is non-increasing. Reporting takes additional $O(K)$ time if there are K answers.*

Assume we have N objects $Q \in \mathcal{Q}$ instead of only one and we want to decide whether there occurs any collision between any pair Q, S, for $Q \in \mathcal{Q}$ and $S \in \mathcal{S}$. We apply the solution to the on-line problem and query the data structure of theorem 1 with each element in $\mathcal{Q}$.

Using this approach we get the following result.

Corollary 1 *Given a set $\mathcal{S}$ of N objects and a set $\mathcal{Q}$ of N objects. Assume that there is a linearization of the collision problem for $\mathcal{Q}$ with respect to $\mathcal{S}$ in the form of (2). Then we can solve in $O(N^{\frac{2dim}{dim+1}+\epsilon})$ time and space the problem of collision detection between any elements of $\mathcal{Q}$ and $\mathcal{S}$.*

We have reduced the collision problem to the task of the formulation of an appropriate linearization. In [6] the linearization is derived from an explicit computation of the collision times. We could do this because the equations of the motions had degree at most two. The natural question is how we can proceed if the motion of the objects is more complicated, i.e. if the equations have degree greater than five (then no explicit formulation of the roots exists).

Actually we do not need an explicit representation of the collision times for the linearization. We only need to know whether two particular objects collide during the specified time period.

4 Collision of two molecules/polyhedra

In the following we deal either with the collision between two molecules or between two polyhedra.

A collision between two molecules occurs if a sphere of one molecule collides with a sphere of the other one.

A collision between two polyhedra is a little bit more difficult to characterize. A collision occurs if a vertex of one polyhedron hits a vertex/edge/face of the other one or if two edges collide. We want to derive polynomial formulas for the different types of collision on the condition that each point $\mathbf{x}$ of body B_i moves according to equation 1. We begin with the discussion of the collision of two spheres. After that we derive a necessary condition for the collision of two edges. By extending the edges to infinite lines we get a set of potential collision times as roots of a univariate polynomial. With the help of additional polynomial inequalities we can restrict this set to those roots which actually represent a collision of the edges (see 4.2-4.3). In order to take care of the restricted duration of the motion we introduce the inequality $g_0(t) := t(1-t) > 0$.

In sections 4.4 and 4.5 we proceed in an analogous way in order to deal with the collision of a vertex and a face. The extension of the face to a plane enables us to find a superset of the desired collision times.

[1] Actually we use only halfspace composition

4.1 Collision of two spheres

If a moving sphere $S_a(t)$ (center $\mathbf{a}$, radius a_0) collides with an other moving sphere S_b it holds:

$$\begin{aligned} & |\mathbf{a}(t) - \mathbf{b}(t)| = a_0 + b_0 \\ \iff & (\mathbf{R}_1(t)\mathbf{a} + \mathbf{o}_1(t) - \mathbf{R}_2(t)\mathbf{b} - \mathbf{o}_2(t))^2 = (a_0 + b_0)^2 \\ \iff & (\mathbf{o}_1(t) - \mathbf{o}_2(t))^2 - 2\mathbf{a}^T\mathbf{R}_1(t)^T\mathbf{R}_2(t)\mathbf{b} \\ & +2(\mathbf{o}_1(t) - \mathbf{o}_2(t))^T(\mathbf{R}_1(t)\mathbf{a} - \mathbf{R}_2(t)\mathbf{b}) \\ & -(a_0 + b_0)^2 + \mathbf{a}^2 + \mathbf{b}^2 = 0 \end{aligned}$$

By multiplying with the denominators we get a polynomial $f(t)$ of degree $2(d_1' + d_2') + 2\max\{d_1, d_2\}$. Its zeros correspond to the collision times.

4.2 Collision of lines

If a line $L_{ab}(t)$ collides with an other line $L_{cd}(t)$ the points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$ lie in a common plane. This can be expressed as the vanishing of the following determinant

$$\begin{aligned} & \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mathbf{a}(t) & \mathbf{b}(t) & \mathbf{c}(t) & \mathbf{d}(t) \end{bmatrix} = 0 \\ \iff \quad & (\mathbf{d}(t) - \mathbf{c}(t))^T(\mathbf{a}(t) \times \mathbf{b}(t)) \\ & +(\mathbf{c}(t) \times \mathbf{d}(t))^T(\mathbf{b}(t) - \mathbf{a}(t)) = 0 \\ \iff \quad & (\mathbf{d} - \mathbf{c})^T\mathbf{R}_2(t)^T\mathbf{R}_1(t)(\mathbf{a} \times \mathbf{b}) \\ & +(\mathbf{c} \times \mathbf{d})^T\mathbf{R}_2(t)^T\mathbf{R}_1(t)(\mathbf{b} - \mathbf{a}) \\ & +(\mathbf{o}_1(t) - \mathbf{o}_2(t))^T(\mathbf{R}_1(t)(\mathbf{b} - \mathbf{a}) \\ & \times\mathbf{R}_2(t)(\mathbf{d} - \mathbf{c})) = 0 \end{aligned}$$

Here the resulting polynomial $f(t)$ has degree $2(d_1' + d_2') + \max\{d_1, d_2\}$.

4.3 Collision of line segments

A collision between two open line segments $l_{ab}(t)$ and $l_{cd}(t)$ (not involving one of the end points) occurs only if the corresponding lines collide and if the following inequalities are fulfilled for $\mathbf{s} = \mathbf{e}_1$ or $\mathbf{s} = \mathbf{e}_2$ or $\mathbf{s} = \mathbf{e}_3$.

$$\begin{aligned} & D_s(\mathbf{b} - \mathbf{a}, \mathbf{c})(t) \ < \ D_s(\mathbf{b}, \mathbf{a})(t) \ < \ D_s(\mathbf{b} - \mathbf{a}, \mathbf{d})(t) \\ \wedge \ & D_s(\mathbf{d} - \mathbf{c}, \mathbf{a})(t) \ < \ D_s(\mathbf{d}, \mathbf{c})(t) \ < \ D_s(\mathbf{d} - \mathbf{c}, \mathbf{b})(t) \\ \vee \ & D_s(\mathbf{b} - \mathbf{a}, \mathbf{c})(t) \ > \ D_s(\mathbf{b}, \mathbf{a})(t) \ > \ D_s(\mathbf{b} - \mathbf{a}, \mathbf{d})(t) \\ \wedge \ & D_s(\mathbf{d} - \mathbf{c}, \mathbf{a})(t) \ > \ D_s(\mathbf{d}, \mathbf{c})(t) \ > \ D_s(\mathbf{d} - \mathbf{c}, \mathbf{b})(t) \end{aligned}$$

with the abbreviation

$$D_s(\mathbf{u}, \mathbf{v})(t) \ = \ \det[\mathbf{s}, \mathbf{u}(t), \mathbf{v}(t)]$$

We want to examine the first inequality in the first row. Substitution of the motion equation for the points yields:

$$\begin{aligned} & \mathbf{s}^T(\mathbf{R}_1(t)(\mathbf{b} - \mathbf{a}) \times (\mathbf{R}_2(t)\mathbf{c} + \mathbf{o}_2(t)) \\ & +\mathbf{R}_1(t)(\mathbf{a} \times \mathbf{b}) + \mathbf{o}_1(t) \times \mathbf{R}_1(t)(\mathbf{b} - \mathbf{a})) < 0 \end{aligned}$$

4.4 Collision of a point and a plane

Since all points of a moving plane $H(t) = \{\mathbf{x} \mid \mathbf{n}(t)^T\mathbf{x} = n_0(t)\}$ fulfill equation (1), the normal vector $\mathbf{n}(t)$ and the parameter $n_0(t)$ change as follows:

$$\begin{aligned} \mathbf{n}(t) \ &= \ \mathbf{R}_i(t)\mathbf{n} \\ n_0(t) \ &= \ n_0 + \mathbf{o}_i(t)^T\mathbf{n}(t) \end{aligned}$$

If a point $\mathbf{a}(t)$ hits the plane $H(t)$ it holds:

$$\begin{aligned} & \mathbf{n}(t)^T\mathbf{a}(t) = n_0(t) \quad \iff \\ & \mathbf{n}^T\mathbf{R}_2(t)^T\mathbf{R}_1(t)\mathbf{a} + \mathbf{n}^T\mathbf{R}_2(t)^T(\mathbf{o}_1(t) - \mathbf{o}_2(t)) = n_0 \end{aligned}$$

4.5 Collision of a point and a triangle

A moving point $\mathbf{a}(t)$ hits the interior of a moving triangle $\Delta_{bcd}(t)$, if it collides with the supporting plane $H_n : \mathbf{n}^T\mathbf{x} = n_0$ of this triangle.

$$\begin{aligned} \mathbf{n} \ &= \ \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{b} \\ n_0 \ &= \ \mathbf{b}^T(\mathbf{c} \times \mathbf{d}) \end{aligned}$$

For $\mathbf{s} = \mathbf{e}_1$ or $\mathbf{s} = \mathbf{e}_2$ or $\mathbf{s} = \mathbf{e}_3$ the following inequalities must additionally hold:

$$\begin{aligned} & D_s(\mathbf{d}, \mathbf{b})(t) \ > \ D_s(\mathbf{a}, \mathbf{b} - \mathbf{d})(t) \ > \ D_s(\mathbf{c}, \mathbf{b} - \mathbf{d})(t) \\ \wedge \ & D_s(\mathbf{d}, \mathbf{c})(t) \ < \ D_s(\mathbf{a}, \mathbf{c} - \mathbf{d})(t) \ < \ D_s(\mathbf{b}, \mathbf{c} - \mathbf{d})(t) \\ \vee \ & D_s(\mathbf{d}, \mathbf{b}(t)) \ < \ D_s(\mathbf{a}, \mathbf{b} - \mathbf{d})(t) \ < \ D_s(\mathbf{c}, \mathbf{b} - \mathbf{d})(t) \\ \wedge \ & D_s(\mathbf{d}, \mathbf{c})(t) \ > \ D_s(\mathbf{a}, \mathbf{c} - \mathbf{d})(t) \ > \ D_s(\mathbf{b}, \mathbf{c} - \mathbf{d})(t) \end{aligned}$$

4.6 Collision of a point and a line

A moving point $\mathbf{a}(t)$ collides with a moving line $L_{cd}(t)$ iff

$$\begin{aligned} & u_1(t)^2 + u_2(t)^2 + u_3(t)^2 = 0 \\ & \mathbf{u}(t) = (\mathbf{d}(t) - \mathbf{c}(t)) \times \mathbf{a}(t) + \mathbf{c}(t) \times \mathbf{d}(t) \end{aligned}$$

4.7 Collision of a point and a line segment

A moving point $\mathbf{a}(t)$ collides with a moving line segment $l_{cd}(t)$ if it collides with the line $L_{cd}(t)$ and the following conditions hold:

$$\begin{aligned} & c_1(t) < a_1(t) < d_1(t) \ \vee \ c_1(t) > a_1(t) > d_1(t) \\ \vee \ & c_2(t) < a_2(t) < d_2(t) \ \vee \ c_2(t) > a_2(t) > d_2(t) \\ \vee \ & c_3(t) < a_3(t) < d_3(t) \ \vee \ c_3(t) > a_3(t) > d_3(t) \end{aligned}$$

4.8 Collision of two points

Two moving points $\mathbf{a}(t)$ and $\mathbf{b}(t)$ collide iff

$$\begin{aligned} & u_1(t)^2 + u_2(t)^2 + u_3(t)^2 = 0 \\ & \mathbf{u}(t) = \mathbf{b}(t) - \mathbf{a}(t) \end{aligned}$$

5 Existence of real roots of a polynomial satisfying several polynomial inequalities

Wanted:

$$\#\{t \in \mathbb{R} \mid f(t) = 0 \wedge g_1(t) > 0 \wedge \ldots \wedge g_l(t) > 0\},$$
where $f(t), g_i(t)$ are polynomials

$Z_f = \{t \in \mathbb{R} \mid f(t) = 0\}$ denotes the set of real roots of polynomial f and $\chi_g(t)$ the characteristic function of the predicate $[g_1(t) > 0 \wedge \ldots \wedge g_l(t) > 0]$. It holds:

$$\begin{aligned} & \#\{t \in \mathbb{R} \mid f(t) = 0 \wedge g_1(t) > 0 \wedge \ldots \wedge g_l(t) > 0\} \\ = & \sum_{t \in Z_f} \chi_g(t) \end{aligned}$$

$\chi_g(t)$ can be expressed as follows

$$\begin{aligned} \chi_g(t) &= 2^{-l} \prod_{i=1}^{l} (1 + \operatorname{sgn} g_i(t)) \\ &= 2^{-l} \sum_{I \in 2^{\{1,\ldots,l\}}} \prod_{i \in I} \operatorname{sgn} g_i(t) \end{aligned}$$

This implies, that

$$\begin{aligned} & \sum_{t \in Z_f} \chi_g(t) = 2^{-l} \sum_{I \in 2^{\{1,\ldots,l\}}} \sum_{t \in Z_f} \prod_{i \in I} \operatorname{sgn} g_i(t) \\ & = 2^{-l} \sum_{I \in 2^{\{1,\ldots,l\}}} \Big(\#\{t \in Z_f | g_I(t) > 0\} \\ & \qquad -\#\{t \in Z_f | g_I(t) < 0\}\Big) \end{aligned}$$
where $g_I(t) = \prod_{i \in I} g_i(t)$ and $g_\emptyset(t) = 1$.

This method for the calculation of $\#\{t \in \mathbb{R} \mid f(t) = 0 \wedge g_1(t) > 0 \wedge \ldots \wedge g_l(t) > 0\}$ goes back to [5]. In this way the original problem is reduced to the calculation of the number of real roots of $f(t)$, which satisfy a single polynomial inequality $[g(t) > 0]$.

6 Hermite's method for the calculation of $\#\{t \in \mathbb{R} \mid f(t) = 0 \wedge g(t) > 0\}$

Let us consider two polynomials

$$\begin{aligned} f(t) &= u_0 t^n + u_1 t^{n-1} + \ldots + u_n \quad \text{and} \\ g(t) &= v_0 t^m + v_1 t^{m-1} + \ldots + v_m \end{aligned}$$

Let $\lambda_1, \ldots, \lambda_n$ denote the roots of $f(t)$ and s_k the Newton sum $\sum_{i=1}^n \lambda_i^k$. In addition we define $h_k = \sum_{i=1}^n g(\lambda_i)\lambda_i^k$. Since s_k and h_k are symmetrical polynomials in $\lambda_1, \ldots, \lambda_n$ they can be expressed as rational functions in the coefficients of f and g. It holds:

$$\begin{aligned} s_k &= \begin{cases} n & \text{for } k = 0 \\ -\dfrac{u_1}{u_0} & \text{for } k = 1 \\ -\dfrac{u_1 s_{k-1} + \ldots + u_{k-1}s_1 + k u_k}{u_0} & \text{for } 2 \le k \le n \\ -\dfrac{u_1 s_{k-1} + u_2 s_{k-2} + \ldots + u_n s_{k-n}}{u_0} & \text{for } k > n \end{cases} \\ h_k &= v_0 s_{k+m} + v_1 s_{k+m-1} + \ldots + v_m s_k \end{aligned}$$

In the following the two Hankel matrices $\mathbf{S}$ and $\mathbf{H}$ play a decisive role. They are composed of the values s_k and h_k.

$$\begin{aligned} \mathbf{S} &= [s_{i+j}]_{i,j=0}^{n-1} = \begin{bmatrix} s_0 & s_1 & s_2 & \ldots & s_{n-1} \\ s_1 & s_2 & s_3 & \ldots & s_n \\ \vdots & \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & s_{n+1} & \ldots & s_{2n-2} \end{bmatrix} \\ \mathbf{H} &= [h_{i+j}]_{i,j=0}^{n-1} \end{aligned}$$

Let S_i (and analogous H_i)

$$S_i = \det \begin{bmatrix} s_0 & s_1 & \ldots & s_{i-1} \\ s_1 & s_2 & \ldots & s_i \\ \vdots & \vdots & & \vdots \\ s_{i-1} & s_i & \ldots & s_{2i-2} \end{bmatrix}.$$

The following theorems hold:

Theorem 2 (Jacobi) *The number of distinct roots of $f(t)$ equals the rank r of matrix $\mathbf{S}$ and the number of distinct real roots equals $r - 2V(1, S_1, S_2, \ldots, S_r)$, where the function $V(.)$ counts the number of sign changes of a sequence.*

Theorem 3 (Hermite,Sylvester) *The number of distinct real roots of $f(t)$ satisfying the condition $g(t) > 0$ equals $r - V(1, S_1, S_2, \ldots, S_r) - V(1, H_1, H_2, \ldots, H_r)$.*

If there are no degeneracies the rank r equals n. If one of the sequences $S_1, \ldots, S_r$ respectively $H_1, \ldots, H_r$ contains zeros, the rule of Frobenius (see [3]) can be applied.

S_i and H_i are rational functions in the coefficients of f and g with denominators u_0^{2i-2} and u_0^{2i+m-2} respectively. The degree of their numerator fulfills: $\deg(\text{numer}(S_i)) = 2i - 2$ and $\deg(\text{numer}(H_i)) = 2i + m - 2$. Especially it holds (see [7]):

$$\begin{aligned} S_n &= \mathcal{D}(f)/u_0^{2n-2} \\ H_n &= \mathcal{R}(f,g)\mathcal{D}(f)/u_0^{2n+m-2} \end{aligned}$$

where $\mathcal{R}(f,g)$ denotes the resultant of f and g and $\mathcal{D}(f) = (-1)^{n(n-1)/2}\mathcal{R}(f,f')$ the discriminant of f. It holds:

$$\mathcal{R}(f,g) = \det \begin{bmatrix} u_0 & u_1 & \dots & u_n & & & \\ & u_0 & u_1 & \dots & u_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & u_0 & u_1 & \dots & u_n \\ v_0 & v_1 & \dots & v_m & & & \\ & v_0 & v_1 & \dots & v_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & v_0 & v_1 & \dots & v_m \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \\ \vdots \\ \leftarrow m \\ \leftarrow m+1 \\ \\ \vdots \\ \leftarrow m+n \end{matrix}$$

7 Linearization

S_i and H_i are rational functions of the coefficients of f and g. S_i can be expressed as follows:

$$S_i = \sum_{\substack{\nu_1,\dots,\nu_n \\ \sum \nu_j = 2i-2}} \alpha^{(i)}_{\nu_1,\dots,\nu_n} \prod_{j=1}^{n} u_j^{\nu_j} / u_0^{2i-2}$$

Since we are only interested in $\operatorname{sign}(S_i)$ it suffices to examine the numerator $\operatorname{numer}(S_i)$ because $u_0^{2i-2} > 0$.

The coefficients u_j can be rewritten as linear functions in d variables $x_1, \dots, x_d$ which uniquely specify the corresponding object. We call these the *linearized coordinates* of the object (E.g. for a sphere with center $\mathbf{a}$ and radius a_0 we set $x_i = a_i$ $(i = 1, \dots, 3)$, $x_4 = a_0$ and $x_5 = a_0^2 + \mathbf{a}^2$. For a line we use its Plücker coordinates.)

$$u_j(x_1,\dots,x_d) = \sum_{k=1}^{d} \gamma_{jk} x_k$$

This substitution transforms the numerator of S_i into the following form.

$$\operatorname{numer}(S_i) = \sum_{\substack{\mu_1,\dots,\mu_d \\ \sum \mu_k = 2i-2}} \zeta^{(i)}_{\mu_1,\dots,\mu_d} \prod_{k=1}^{d} x_k^{\mu_k}$$

If we introduce a new variable y_j for each product of the form $\prod_{k=1}^{d} x_k^{\mu_k}$ with $\sum_{k=1}^{d} \mu_k = 2i-2$ we get a linearized expression for $\operatorname{numer}(S_i)$ with at most $\binom{2i-d-3}{d-1}$ terms.

$$\sum_{j=1}^{\binom{2i-d-3}{d-1}} \zeta_j y_j$$

Analogously we get a linearized expression for $\operatorname{numer}(H_i)$ with at most $\binom{2i+m-d-3}{d-1}$ terms.

$$\sum_{j=1}^{\binom{2i+m-d-3}{d-1}} \xi_j z_j$$

where z_j is of the form $\prod_{k=1}^{d} x_k^{\mu_k}$ with $\sum_{k=1}^{d} \mu_k = 2i + m - 2$.

Whether the predicate $[\#\{t \in Z_f | g_1(t) > 0 \wedge \dots \wedge q_l(t) > 0\} > 0]$ is true or not only depends on the signs of the expressions S_i and $H_i(g_I)$ for $I \in 2^{\{1,\dots,l\}}$. Therefore we can write it as a $\{0,1\}$-valued function $P(.)$ of these signs. I.e. we can arrange all possible sign values for S_i and $H_i(g_I)$ in a function table with $n2^l$ input columns. If we express the function value in disjunctive normal form we get

$$\#\{t \in Z_f | g_1(t) > 0 \wedge \dots \wedge g_l(t) > 0\} > 0$$
$$\iff \bigvee_{\varepsilon \in \{-1,0,1\}^{n2^l}} P(\varepsilon) \bigwedge_{j=1}^{n2^l} \operatorname{sign}(X_j) = \varepsilon_j$$

where each X_j corresponds to $\operatorname{numer}(S_i)$ or $\operatorname{numer}(H_i(g_I))$.

Lemma 1 *We get a linearization of dimension* $dim = \binom{2n+m+d-3}{d-1}$ *consisting of* $dis = O(3^{n2^l})$ *disjunctions and* $con = O(n2^l)$ *conjunctions. Thereby n denotes the degree of the polynomial f, l the number of inequalities g_i, m the maximal degree of all polynomials g_I, and d the number of the linearized coordinates of the involved objects.*

8 Summary

In our model of computation we only assume that the basic arithmetic operations $+$, $-$, and $*$ can be done in constant time. For a given problem instance the terms *dim*, *dis* and *con* are constants. Therefore we can combine Lemma 1 with Theorem 1 and get the following result.

Theorem 4 *The collision detection problem for two polyhedra or molecules, whose trajectories are described by polynomials, can be solved in subquadratic time.*

A Appendix

A.1 Orientations Described by Quaternions

The orientation of a rigid body in 3-space can be described by a quaternion

$$\mathfrak{r} = \begin{bmatrix} r_0 \\ \mathbf{r} \end{bmatrix} \in \mathbb{R}^4.$$

This orientation results from a rotation of the world frame about the axis with direction $\mathbf{r} \in \mathbb{R}^3$. The rotation angle is determined by $|\mathbf{r}|$ and the scalar r_0.

The 0-th component of a quaternion is called the scalar part, the other components comprise the vector part. Vectors in 3-space are interpreted as quaternions with scalar part 0. Quaternions form a vector space with an associative multiplication defined by

$$\mathfrak{q} \cdot \mathfrak{p} \;=\; \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} q_0 p_0 - \mathbf{q}^T \mathbf{p} \\ q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p} \end{bmatrix}.$$

The quaternion product is linear in $\mathfrak{p}$ and $\mathfrak{q}$. The conjugate quaternion $\mathfrak{q}^*$ of $\mathfrak{q}$ is formed by negating the vector part of $\mathfrak{q}$. The product $\mathfrak{q} \cdot \mathfrak{q}^*$ yields the scalar value $q_0^2 + \mathbf{q}^2$, which corresponds to the length of $\mathfrak{q}$ under the Euclidean metric in $\mathbb{R}^4$. Quaternions which satisfy $\mathfrak{q} \cdot \mathfrak{q}^* = 1$ are called unit quaternions. Let $\mathfrak{r}$ be a given quaternion. Then the mapping

$$\mathfrak{a} = \begin{bmatrix} 0 \\ \mathbf{a} \end{bmatrix} \;\rightarrow\; \mathfrak{a}' = \begin{bmatrix} 0 \\ \mathbf{a}' \end{bmatrix} = \frac{\mathfrak{r} \cdot \mathfrak{a} \cdot \mathfrak{r}^*}{\mathfrak{r} \cdot \mathfrak{r}^*}$$

describes a rotation of the vector $\mathbf{a}$ about the axis $\mathbf{r}$ about the angle $\varphi = 2\arctan(|\mathbf{r}|/r_0)$. In matrix notation, this amounts to $\mathbf{a}' = \mathbf{R}(\mathfrak{r}) \cdot \mathbf{a}$, with

$$\mathbf{R}(\mathfrak{r}) \;=\; \frac{(r_0^2 - \mathbf{r}^2)\mathbf{I} + 2\mathbf{r}\mathbf{r}^T + 2r_0\mathbf{r}^{\times}}{r_0^2 + \mathbf{r}^2}. \qquad (3)$$

Here, $\mathbf{r}^{\times}$ denotes the canonical skew-symmetric matrix corresponding to $\mathbf{r}$. It can be easily verified that $\mathbf{R}(\mathfrak{r}) \cdot \mathbf{R}(\mathfrak{r})^T = \mathbf{I}$ and $\det(\mathbf{R}(\mathfrak{r})) = 1$.

Suppose the orientation $\mathfrak{r}(t)$ of a rigid body depends on a time parameter t in the following way

$$\mathfrak{r}(t) \;=\; \mathfrak{p} + t(\mathfrak{q} - \mathfrak{p}) \quad \text{for } t \in [0,1].$$

The motion of the body induced by this varying orientation is a simple rotation about an axis the direction of which is given by the vector part of the quaternion product of $\mathfrak{q} \cdot \mathfrak{p}^*$. This axis of rotation can be determined by calculating the instantaneous angular velocity $\omega(t)$ of the vector $\mathbf{a}(t) = \mathbf{R}(\mathfrak{r}(t)) \cdot \mathbf{a}$.

$$\begin{aligned} \frac{d\mathbf{a}(t)}{dt} &= \omega(t) \times \mathbf{a}(t) = \omega^{\times}(t) \cdot \mathbf{a}(t), \\ &= \frac{d\mathbf{R}(t)}{dt} \cdot \mathbf{a} = \omega^{\times}(t)\mathbf{R}(t) \cdot \mathbf{a} \end{aligned}$$

This implies that

$$\omega^{\times}(t) \;=\; \frac{d\mathbf{R}(t)}{dt} \cdot \mathbf{R}(t)^T.$$

It turns out that the direction of $\omega(t)$ is time invariable and only its magnitude changes with time.

References

[1] M. Ben-Or, D. Kozen, and J. Reif: *The complexity of elementary algebra and geometry*, J. Comp. and Sys. Sci., Vol. 32: 251–264, (1986)

[2] J. Canny: *An improved sign determination algorithm*, AAECC-91 (1991), New Orleans

[3] F.R. Gantmacher: *The Theory of Matrices*, Chelsea, New York, (1959)

[4] M. van Kreveld: *New Results on Data Structures in Computational Geometry*, Ph.D. Thesis, University of Utrecht, The Netherlands, (1992)

[5] P. Pedersen: *Counting real zeros*, New York University, NYU Technical Report 545-R243, (1991)

[6] E. Schömer and C. Thiel: *Efficient collision detection for moving polyhedra*, Proc. 11th Annu. ACM Sympos. Comput. Geom.: 51–60, (1995)

[7] A. Y. Uteshev and S. G. Shulyak: *Hermite's method of seperation of solutions of systems of algebraic equations and its applications*, Linear Algebra and its Applications 177: 49–88, (1992)