Direction Weighted Shortest Path Planning

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Abstract

We examine shortest path planning under a direction weighted Euclidean metric, motivated by the problem of finding optimal routes for sailboats. We show that for piecewise linear weight functions the shortest path in the unrestricted plane always consists of 2 *line segments, and by this solve the problem in a polygonal environment with a visibility graph approach. We then investigate the problem in a general setting, in which the weighted distance measure is supplemented by a link distance measure, and present polynomial approximation algorithms for this case.*

1 Introduction

Shortest path problems belong to the classical topics in computational geometry and motion planning. Beside the problem of finding a path with minimal length according to the Euclidean metric, different optimization criteria are studied in the literature, e.g. the number of line segments (*link distance*) or the sum of all turning angles (*total turn*) of a polygonal path [5, 1, 2].

Generalizations of the Euclidean length shortest path problem also include the weighting of the metric. In the *weighted region problem* the task is, given a polygonal subdivision of the plane into constantly weighted regions, to find a shortest path according to the regionally weighted metric [4].

We examine shortest path planning under a *direction weighted* Euclidean metric. This is motivated by the problem of finding optimal routes for sailboats, further denoted as the *sailor's problem* (see [7] for different aspects of vessel routing).

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1.1 Basic definitions

The speed of a sailboat depends on the direction of the wind, the position of the sail, and its course. If we assume the wind direction to be fixed, and the position of the sail as optimal for the current course, then the speed is a function of the course. If we ask for time-optimal paths, this relationship corresponds to a weighting of the Euclidean distance by a continuous function $\omega : [-\pi, \pi] \to (0, \infty)$. This function may have several local minima, corresponding to sailing courses close to the wind and down wind.

In practice, a sailing route is a finite sequence of straight line courses. Generally, each change in the course is a time-consuming turn, at which the sail is to be trimmed again. We thus may wish to fine turns by a fixed constant L.

Let $v_1, \ldots, v_n$ be vertices in the plane, and let $p = (v_1, \ldots, v_n)$ be the path consisting of the line segments (vectors) $a_i = v_i\vec{v_{i+1}}$, $i = 1, \ldots, n-1$. Let $\alpha_i = \angle a_i$ be the angle between the positive x-axis and the vector a_i, and let $|a_i|_e$ be the Euclidean length of the segment a_i. W.l.o.g. we assume $\alpha_i \neq \alpha_{i+1}$ for all $i = 1, \ldots, n-2$. Then we define the (ω, L)-length of p to be

$$|p|_{\omega,L} := \sum_{i=1}^{n-1} \omega(\alpha_i) \cdot |a_i|_e + (n-2)L.$$

We use $|p|_\omega$ short for $|p|_{\omega,0}$, and e for the constant weight function $e \equiv 1$. Thus, $|p|_e$ corresponds to the Euclidean length of p.

Assume we are given an environment $E \subseteq \mathbf{R}^2$, a start point s and a target point $t \in E$. Let $\Gamma(s,t,E)$ be the set of all paths $p = (v_1, \ldots, v_n)$ in E with $v_1 = s$, $v_n = t$ and $n \in \mathbf{N}$. We define the (ω, L)-distance by

$$d_{\omega,L}(s,t,E) := \inf\{\ |p|_{\omega,L}\ ;\ p \in \Gamma(s,t,E)\ \}.$$

Note that for $L = 0$ the minimum of the above set does not necessarily exist (i.e., there may be no optimal path that is polygonal). The (approximate) sailor's problem is, given s, t, E and ε, to construct a polygonal path p from s to t in E with

$$|p|_{\omega,L} \leq (1+\varepsilon) d_{\omega,L}(s,t,E).$$

In this paper, we investigate the complexity of several instances of the sailor's problem, including the case of E being a polygonal environment and $L > 0$.

1.2 Related work

Compared to the intensively studied weighted region problem, the problem of sailing may be considered as an alternative generalization of the Euclidean shortest path problem. The weighted region problem can be solved by an approximation algorithm in polynomial time [4]. The algorithm is based on a continuous Dijkstra technique. However, the comparable case in our problem ($L = 0$) turns out to be simpler.

The more interesting case of $L > 0$ is closely related to bicriteria shortest path problems as examined in [1]. This case is a direct extension to the planning problem for a cost function that combines (sums up) Euclidean length and number of turns (resp., links), which itself can be considered as a generalization to the k-link shortest path problem [2].

While the link distance between 2 points in a polygonal environment can be computed in polynomial time [5], the problem of finding an Euclidean shortest path with $\leq k$ links seems to be significantly harder: for simple polygons, there exists an ε-approximation algorithm. However, in polygons with holes only a constant factor approximation algorithm is known. The problem of sailing adds even more combinatorial structure to this problem. Note e.g. that shortest paths may consist of exponentially many segments (this case arises if we have to 'cruise' inside a narrow long channel).

1.3 Summary of main results

In section 2, we investigate the sailor's problem for $L = 0$. We focus on the case of ω being piecewise linear. Based on several geometrical lemmata, we show that the (ω, L)-shortest path in an unrestricted environment consists of ≤ 2 line segments (theorem 1). This result can be extended to the case of E being restricted by polygonal obstacles. The ω-shortest path can then be planned as a zig-zag path that approximates the shortest path in a weighted visibility graph (theorem 2).

In section 3 we address the case of $L > 0$. We present an algorithm which approximates the (ω, L)-shortest path in polynomial time under a reasonable set of additional conditions (theorem 4). The algorithm computes a proper lattice of path segment candidates and thus reduces the problem to a graph searching problem, which can be solved by a modified

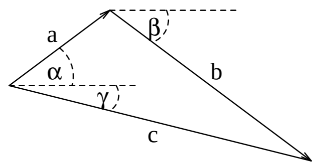

figure 1

Dijkstra algorithm. The additional conditions are due to a tradeoff between the robustness of the path and its length (similar tradeoffs are also used in approximation techniques for optimal kinodynamic motion planning [3]). Without this tradeoff, we get a constant factor approximation (theorem 3).

2 Direction Weighted Shortest Paths in the Plane

In the introduction, we have formulated the sailor's problem in a discrete way by confining to polygonal paths. For the case $L = 0$ however, it stands to reason to consider the class of all (piecewise) smooth paths $p : [0, 1] \to \mathbf{R}^2$.

But at least for the case of ω being piecewise linear, we can guarantee that the ω-distance between 2 points in $E = \mathbf{R}^2$ is always reached by a finite sequence of line segments:

A basic observation concerning weighted distances is that the triangle inequality respective to ω (in figure 1: $|a|_\omega + |b|_\omega \geq |c|_\omega$) does not necessarily hold. However, if ω is linear during the interval $I = [\gamma_1, \gamma_2]$, then a curve discussion shows that the triangle inequality holds *locally*, i.e. for all angles (in figure 1: α, β, γ) ranging over I. This induces, that any ω-shortest path p between s and t can locally be substituted by straight line segments connecting intermediate points of p, q.e.d.

In fact, we believe that the above claim holds for any smooth, positive weight function. But in the following, let us assume that ω is piecewise linear, and given fixed with minimum > 0. (For simplicity, we do not consider ω as input. In fact, minimal and maximal slope of ω determine our constants.)

As the next 2 lemmata will reveal, the ω-shortest connection between 2 points in the unrestricted plane can always be chosen as a path consisting of not more than 2 line segments.

Lemma 1 *If $s, t \in \mathbf{R}^2$ can be connected by a polygonal path p, then they can also be connected by a polygonal path p' with $|p'|_\omega = |p|_\omega$, whose segments lie each on the same side of the line $\vec{st}$, and touch $\vec{st}$ with one of their endpoints.*

Proof (Sketch): Each polygonal path p =

$(v_1, \ldots, v_n)$ can be interpreted as a sequence of vectors $a_i = \vec{v_i v_{i+1}}$, $i = 1, \ldots, n-1$, starting at s. By breaking these vectors into parts and rearranging them, we get paths from s to t of equal ω-length. With this observation, it is trivial to construct a path p' meeting the requirements, with at most double the number of segments as p. $\square$

Lemma 2 *Let p be a path from s to t, defined by the sequence of vectors a_1, b_1, a_2, b_2, with $s + a_1 + b_1 \in \vec{st}$ and $\angle a_i \geq \angle \vec{st}$, $i = 1, 2$.*
Then there exists a path p' connecting s and t by 2 segments, with $|p'|_\omega = |p|_\omega$.

Proof (Sketch): Let $u = s + a_1 + b_1$, $c_1 = \vec{su}$, and $c_2 = \vec{ut}$. By a simple calculation, we get

$$(|a_i|_\omega + |b_i|_\omega)/|c_i|_e = const_i,$$

with $const_i$ only depending on $\alpha_i = \angle a_i$ and $\beta_i = \angle b_i$, but not on the Euclidean length of the vectors.
W.l.o.g. let $const_1 \leq const_2$, and p' be the path obtained by scaling a_1 and b_1 such that $u = t$. Then

$$\begin{aligned} |p'|_\omega &= const_1 \cdot (|c_1|_e + |c_2|_e) \\ &\leq const_1 |c_1|_e + const_2 |c_2|_e = |p|_\omega. \end{aligned}$$ $\square$

Let p be a polygonal path with minimal ω-length. By constructing a path p' according to lemma 1, and then inductively applying lemma 2, we get a 2-segment connection between s and t. This connection is uniquely defined by a pair of optimal angles $(\alpha_{opt}, \beta_{opt})$.

Our actual goal is to compute good approximate solutions to the ω-shortest path problem. The next lemma shows that the relative error in the weighted path length $|a_{opt}|_\omega + |b_{opt}|_\omega$ and the absolute error in the angles $(\alpha_{opt}, \beta_{opt})$ are polynomially related:

Lemma 3
Let $|\alpha - \gamma| \leq |\alpha_{opt} - \gamma|$ and $|\beta - \gamma| \leq |\beta_{opt} - \gamma|$.
If $|\alpha - \alpha_{opt}| < \delta$ and $|\beta - \beta_{opt}| < \delta$, then

$$|a|_\omega + |b|_\omega \leq (1 + c_\omega \sqrt{\delta}) \cdot (|a_{opt}|_\omega + |b_{opt}|_\omega)$$

for a fixed constant c_ω and δ sufficiently small.

Approximation of a pair of angles (α_{opt},β_{opt}) for given s,t can be done by numerical optimization. For brevity we will denote the time needed to find such angles with absolute error ε as $T(\varepsilon, n)$; n denoting the bit complexity of s and t. With this notation, we get the following 'bang-bang type' theorem:

Theorem 1 *In $E = \mathbf{R}^2$, any 2 points s and t can be connected by a 2-segment path p_{opt} with $|p_{opt}|_\omega = d_\omega(s, t, \mathbf{R}^2)$. The distance $d_\omega(s, t, \mathbf{R}^2)$ can be approximated with a relative error of ε in time $T(O(\varepsilon^2), n)$.*

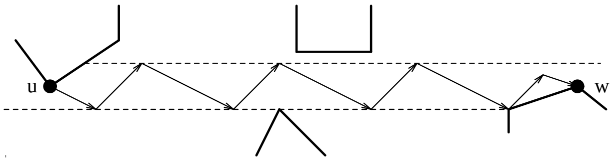

figure 2

Now let us consider the sailor's problem in a polygonal environment $E \subseteq \mathbf{R}^2$. Again, we have to deal with the problem that the ω-distance $d_\omega(s, t, E)$ may be a limit case: figuratively, this happens if the optimal path approaches a vertex at a convex corner of E via a zig-zag path with an infinite number of 'legs'. To rule out such degenerate cases, we require E to be a 2-dimensional closed manifold, and s and t to lie inside the open interior $int(E)$ of E.

The following 2 lemmata reduce the ω-shortest path problem in a polygonal environment to the case of $E = \mathbf{R}^2$. (We shall call 2 points $u, w \in E$ *strongly visible*, if the line through u and w does not penetrate into the obstacle space at u and w.)

Lemma 4 *If u and w are strongly visible points in E, then they can be connected by a polygonal path p with $|p|_\omega = d_\omega(u, w, \mathbf{R}^2)$.*

Proof (Sketch): Consider the line segment l connecting u and w. W.l.o.g. we can assume that l touches no vertex of E except possibly u and w. Let B_ε be a rectangular box around l, whose sides have distance ε to l. Then there exists an ε such that B_ε intersects only edges of E adjacent to u or w. Pick a pair of optimal angles, then inside $E \cap B_\varepsilon$ there must be at least one optimal vector emanating from u and at least one pointing to w (note that this is only true because of the strong visibility of u and w). Using these vectors at an appropriate length as first and last path segments, it is easy to connect u and w inside $E \cap B_\varepsilon$ by an optimal zig-zag path p with finitely many segments (see figure 2). $\square$

Lemma 5 *Let p be any polygonal path in E connecting s and t. Then there exists a polygonal path p' from s to t in E, which has the same ω-length as p and connects vertices of E that are strongly visible.*

Proof (Sketch): W.l.o.g. we can assume that p has no loops. Let q be the Euclidean shortest path from s to t lying in the homotopy class of p. Then q connects strongly visible points of E. By breaking the vectors of p into parts and rearranging them, we can construct a path p' as required above, connecting the same vertices as q. $\square$

The above lemmata show the existence of a polygonal path p with $|p|_\omega = d_\omega(s,t,E)$ (provided E is connected). Moreover, we know that there exists a sequence of strongly visible vertices $v_1, \ldots, v_n$ of E such that

$$d_\omega(s,t,E) = \sum_{i=1}^{n-1} d_\omega(v_i, v_{i+1}, \mathbf{R}^2).$$

Thus we can approximate $d_\omega(s,t,E)$ with a relative error of ε by computing the visibility graph of E, weighting each edge (u,v) by an ε-approximation of the ω-distance $d_\omega(u,v,\mathbf{R}^2)$, and running Dijkstra on the resulting graph.

Theorem 2 *If E is a connected, 2-dimensional manifold, then any 2 points $s,t \in int(E)$ can be connected by a polygonal path p with $|p|_\omega = d_\omega(s,t,E)$.*
The distance $d_\omega(s,t,E)$ can be approximated with a relative error of ε in time $O(n_v^2 T(O(\varepsilon^2), n))$; n denoting the bit complexity of E (including s and t), and n_v the number of vertices.

3 Shortest Paths Under a Combined Link / Distance Cost Measure

The theoretical results on ω-shortest paths of the previous section are based on the fact that path segments can be broken and rearranged without cost, leading to slender zig-zag paths. Returning to the motivation of planning optimal routes for sailboats, it is desirable to fine turns by constant costs. In this section, we develop an approximation technique for the (ω, L)-shortest path problem ($L > 0$) inside polygonal environments.

In the following lemma, we show that we can confine our search to paths of some normal form.

Lemma 6 *If E is connected, then there exists a polygonal path p from s to t in E with $|p|_{\omega,L} = d_{\omega,L}(s,t,E)$ such that each segment of p touches the boundary of E (including s or t).*

Proof (Sketch): Consider the following situation: s and t are connected by a path p consisting of 3 line segments (resp., vectors) a, b and c. Then it can be evaluated that $|p|_\omega = const_1 |a|_e + const_2$, with $const_1$ and $const_2$ only depending on the angles $\alpha = \angle a$, $\beta = \angle b$ and $\gamma = \angle c$.
Let $u = s + a$ be the endpoint of the segment a. Consider the paths p_u that we obtain when we move u up or down on the line through a and connect u to t by the 2 vectors uniquely determined by β and γ. The above equation shows that there is one direction for moving u, in which $|p_u|_\omega$ does not increase.
Now assume that p is (ω, L)-shortest, and b does not touch the boundary of E. Then $|p_u|_\omega$ must be constant, and if we move u to either direction, b will hit the boundary of E before p_u becomes a 2-segment path. □

Note that a path of the described normal form may still contain segments that touch the boundary of E somewhere at an edge, but at no vertex. Our strategy for searching (ω, L)-shortest paths will be as follows:

1. Subdivide the edges of E by a polynomial number of vertices (subdivision vertices).
2. Surround each of the original and subdivision vertices by a 'halo' consisting of a polynomial number of rays.
3. Construct a graph G by intersecting these rays.
4. Perform a modified Dijkstra algorithm to determine the (ω, L)-shortest path in G.

In the first step, we use a technique due to Papadimitriou to get an adequate polynomial subdivision of the edges of E. We take into account that the edges may have exponential length by dividing them into parts of geometrically increasing length.

Let e be the edge under consideration, $x_0 \in e$ the point on e with minimal Euclidean distance to s, and d this distance. Define the vertices x_i and x_{-i}, $i \in \mathbf{N}$, to be the points on e with Euclidean distance $d_i = \delta d(1+\delta)^{i-1}$ to x_0. Then the following holds [6]:

Lemma 7 (Papadimitriou) *Let n be the bit complexity of E (including s and t), n_v the number of original vertices, and n_s the number of the constructed subdivision vertices. Then*
(1) $n_s = O(n_v(n + \log(1/\delta))/\delta)$, and
(2) $|x_i \vec{x}_{i+1}|_e \leq \delta |s\vec{x}_i|_e \leq \delta d_e(s,x,E)$
for all x on the segment $x_i \vec{x}_{i+1}$.

This can be extended to ω-distances as follows:

Lemma 8 *Let a be the line segment (resp., vector) from any point $v \in E$ to any point x on the segment $x_i \vec{x}_{i+1}$, then*

$$|v\vec{x}_j|_\omega \leq |a|_\omega + c_\omega \delta \cdot d_\omega(s,x,E)$$

for $j \in \{i, i+1\}$ and c_ω a fixed constant.

Additional to the subdivision vertices x_i, we take all intersection points between edges and extended visibility lines $r(u,u')$ of obstacle vertices u, u'.

Let E_S denote the 'environment' after the described subdivision. In the second step of the algorithm, we surround each vertex of E_S by a halo of rays at uniformly stepped angles, plus extended visibility lines.

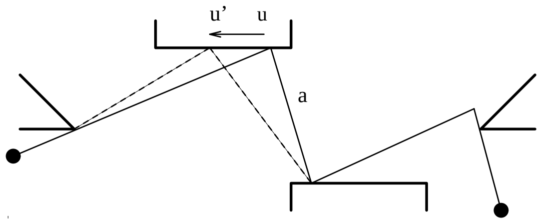

figure 3

For any vertex $u \in E_S$, let $r_\alpha(u)$ denote the ray emanating from u at angle α. Given $\rho = \pi/k$, we define

$$R_\rho := \bigcup_{u \in E_S} (\bigcup_{i=1..k} r_{(i/k)\pi}(u) \cup \bigcup_{u' \neq u \in E_S} r(u, u')).$$

In the third step of the algorithm, we interpret R_ρ as a planar graph G, whose vertices V_G correspond to the intersection points of the rays, and whose edges E_G correspond to the parts of the rays between 2 intersection points. We make G directed by substituting each edge by 2 opposite directed edges. The edges are weighted by their ω-length.

In the last step, we search for the (ω, L)-shortest path in G by reducing the problem to a regular shortest path problem in a weighted (non-planar) graph G':

Lemma 9 *Let $G = (V_G, E_G) \subseteq \mathbf{R}^2$ be an embedded planar digraph, and let k be the total number of the distinct directions of its edges. Then the (ω, L)-shortest path from $s \in V_G$ to $t \in V_G$ in G can be computed in time $O(|E_G| + k|V_G|\log(k|V_G|))$.*

Proof (Sketch): We construct a graph G' which contains one copy of G for each of the k distinct directions $\alpha_1, \ldots, \alpha_k$. Let $V_{G'} = \{ v_i \ ; \ i = 0 \ldots k, \ v \in V_G \}$, $E_1 = \{ (v_0, v_i), (v_i, v_0) \ ; \ i = 1 \ldots k, \ v \in V_G \}$, $E_2 = \{ (v_i, u_i) \ ; \ (v, u) \in E_G \ \wedge \ \angle \vec{vu} = \alpha_i \}$ and $E_{G'} = E_1 \cup E_2$. Define the weight function $w : E_{G'} \to \mathbf{R}$ by $w((v_i, u_i)) = \omega(\alpha_i)$ for all $(v_i, u_i) \in E_2$, $w((v_0, v_i)) = w((v_i, v_0)) = L/2$ for $(v_0, v_i) \in E_1$, $v \neq s, t$, and $w((s_0, s_i)) = w((t_0, t_i)) = w((s_i, s_0)) = w((t_i, t_0)) = 0$. Finally, set the new source and target to be $s' = s_0$ and $t' = t_0$.
Then the shortest path from s' to t' in G' has length $d_{\omega,L}(s, t, G)$, and can be computed with Dijkstra's algorithm in the predicted runtime. □

The set of rays R_ρ and thus also the planar graph G have a number of edges and vertices which is polynomial in $1/\rho$ and n_s, with n_s as in lemma 7. Assuming the time needed to evaluate ω as constant, the total runtime of our algorithm is thus polynomial in $1/\rho$, $1/\delta$ and the input complexity n.

We shall analyze the above algorithm by deformation arguments. Our goal is to show, that the constructed graph contains an 'approximation' of an actual (ω, L)-shortest path. Using lemma 8, we first sharpen our normal form by showing that there exists a "nearly optimal path" close to a shortest such that each segment of this path has an endpoint that coincides with a vertex of E_S.

Lemma 10 *Let p_{opt} be an (ω, L)-shortest path from s to t in E_S with k segments, each of them touching the boundary of E_S. Then there exists a path p from s to t with*

$$|p|_\omega \leq (1 + kc_\omega\delta)|p_{opt}|_\omega$$

and $\leq 2k$ segments, each touching a vertex of E_S.

Proof (Sketch): Let a be a segment of $p := p_{opt}$ not touching a vertex of E_S. Then a touches an edge e of E_S at some point u (figure 3). Let a and a' be the 2 path segments incident to u. As u is no vertex, the segment a' can touch at most one obstacle vertex (else u would lie on a visibility line). There are 2 cases.
First case: if a' touches no vertex, we simultaneously move a and a' by sliding u on e in either direction, while keeping the endpoints of a and a' opposite to u fixed. We do this until either a or a' touches a vertex If a' is this segment, we repeat the process for a.
Second case: if a' touches one vertex, we break a' into 2 parts at this vertex, and move the part a'_u with endpoint u simultaneously with the segment a. We do this until u coincides with a subdivision vertex or until a hits an obstacle vertex.
We repeat the whole process until p has the desired form. Observing that no segment can be broken twice, we get the claimed properties. □

Now consider the path p constructed above. First observe that 2 neighboring rays at a vertex define a 'wedge' (see figure 4). The path p lies in a sequence of wedges $W_1, \ldots, W_k$ at vertices $v_1, \ldots, v_k$, such that the i-th segment a_i of p lies inside W_i and either intersects v_i or touches v_i with an endpoint. Let p'_i denote the Euclidean shortest path from v_i to v_{i+1} in $W_i \cup W_{i+1}$, and let $p' = p'_1 \ldots p'_{k-1}$ be the composition of these paths. Intuitively, the construction of p' corresponds to a *stretching* of p between the vertices v_i (which may be no vertices of p!). Again, this process may lead to the breaking of segments, but no segment of p_{opt} that has been broken during the construction of p will be broken again. Thus p' is a polygonal path with less than $2k$ segments, and lies entirely in R_ρ (resp., G). Further, lemma 3 reveals that $|p'|_\omega \leq (1 + c_\omega\sqrt{\rho})|p|_\omega$.

Assume now that we are given an a priori error bound ε, and a bound k on the number of segments of any feasible (ω, L)-shortest path. The bound k may be implicitly given by L and a bound M on $d_{\omega,L}(s, t, E)$:

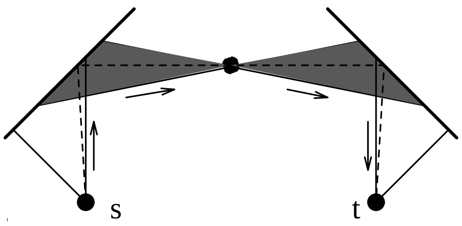

figure 4

as no path with (ω, L)-length $\leq M$ can have more than M/L segments, we can set $k = M/L$.

Choosing $\delta = O(\varepsilon/k)$ and $\rho = O(\varepsilon^2)$, we can show that the graph G contains a path p' with $|p'|_\omega \leq (1+\varepsilon)|p_{opt}|_\omega$, which has at most double the number of segments than p_{opt}. Thus it satisfies $|p'|_{\omega,L} \leq 2d_{\omega,L}(s,t,E)$ for ε sufficiently small.

Theorem 3 *If $d_{\omega,L}(s,t,E) \leq M$, then a path p from s to t inside E fulfilling*

$$|p|_{\omega,L} \leq 2d_{\omega,L}(s,t,E)$$

can be computed in time polynomial in the complexity of E and in $k = M/L$.

By successively probing $k = 1, 2, 3, \ldots$ until $k > |p'_k|_{\omega,L}/L$ (with p'_k the output at the k-th stage), we can actually substitute M by $d_{\omega,L}(s,t,E)$.

In the last part of this section, we discuss how our technique can be made an ε-approximation scheme (note that the idea applies also to the k-link shortest path problem).

By breaking segments, we have introduced additional turns at obstacle vertices. However, these turns are very small, and by replacing a 2-segment connection over such an additional turn by the adequate 1-segment connection, we would only penetrate into the obstacle space by a relatively small amount. We thus relax the described problem by requiring the optimal path to be sufficiently robust.

In the following, we assume E to be bounded. Then let $D := max\{\ |\vec{sx}|_e\ ;\ x \in E\ \}$ denote the 'diameter' of E with respect to s. Let E^ε be any fixed polygonal environment such that

$$E^\varepsilon \subseteq \{\ x \in E\ ;\ dist(x, \mathbf{R}^2 - E) \geq \varepsilon D\ \},$$

with *dist* denoting the Euclidean Hausdorff distance. (We may think of E^ε as a 'shrunk' environment.)

Now let p_{opt} be an (ω, L)-shortest path in E^ε. We run the first 3 steps of our algorithm for this environment, and supplement the computed graph G by additional lines (resp., edges) as 'bridges' to avoid additional turns:

If u_1 and u_2 are vertices of G that lie on 2 rays intersecting at an obstacle vertex v, and if $a = \vec{u_1u_2}$ lies inside E, then we connect u_1 and u_2 by an edge. We denote the resulting graph by $\tilde{G}$.

We know that the graph G contains a path p' with $|p'|_\omega \leq (1+\varepsilon)|p_{opt}|_\omega$, which makes only some additional turns at obstacle vertices more than p_{opt}. With $\delta = O(\varepsilon/k)$ and $\rho = O(\varepsilon^2)$, it is easy to check that these additional turns can be avoided by using the bridges without increasing the ω-length significantly. We can show that $\tilde{G}$ contains a path p'' with $|p''|_{\omega,L} \leq (1+\varepsilon)d_{\omega,L}(s,t,E^\varepsilon)$. This finally proves

Theorem 4 *If $d_{\omega,L}(s,t,E^\varepsilon) \leq M$, then a path p from s to t inside E with*

$$|p|_{\omega,L} \leq (1+\varepsilon)d_{\omega,L}(s,t,E^\varepsilon)$$

can be computed in time polynomial in the complexity of E^ε, in ε and in $k = M/L$.

Note that this theorem does not tell us how to compute a shrunk environment E^ε (methods are straightforward). Analogous to theorem 3, the bound M can be substituted by $d_{\omega,L}(s,t,E^\varepsilon)$.

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