On the Topological Structure of Configuration Spaces

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Abstract. We investigate the topological structure of configuration spaces of kinematic scenes, i.e. mechanisms consisting of rigid, independently movable objects in a 2– or 3–dimensional environment. We demonstrate the practical importance of the considered questions by giving a motivation from the viewpoint of qualitative reasoning. Especially, we investigate topological invariants of the configuration space as a means for characterizing and classifying mechanisms.
This paper focuses on the topological invariants homeomorphy, homotopy equivalence and fundamental group. We describe a procedure for computing a finite representation of the fundamental group of a given kinematic scene, and investigate its possible structure and complexity for simple classes of scenes. We show that for any finitely represented group one can construct a kinematic scene in the plane such that the fundamental group of its configuration space is isomorphic to the given group. This construction shows the undecidability of a variety of problems concerning the topological structure of configuration spaces, and reveals that the considered invariants in their general form do not provide an algorithmic method for characterizing or classifying arbitrary mechanisms.

Key words: qualitative reasoning, qualitative kinematics, mechanisms, configuration space, topology, homeomorphy, homotopy, fundamental group.

1 Introduction

We consider mechanisms consisting of rigid objects in a 2– or 3–dimensional euclidian space, the physical world. We denote a collection of objects as a *kinematic scene* since we will not consider dynamic constraints.
The concept of *configuration space* and *free space* is central for the wide field of robotics. Although both notions are often used synonymous, we will strongly distinguish between them in the following. With "configuration space of a kinematic scene" we denote the topological space which is spanned by all the parameters that determine the positions of the movable objects bijectively. At a fixed time point, the positions of all objects in the scene can thus be described by a point of the configuration space, called a *configuration.*

If we consider as an example two rotary gearwheels in the plane, i.e. a scene consisting of two objects with rotational freedom of motion, then the configuration space of this scene is a torus: The rotation angles of both gearwheels span a rectangle; by identifying opposite sides of this rectangle, corresponding to the rotation angles $0°$ and $360°$ of each gear, the torus is generated.

As the parts of a mechanism generally constrain their positions, the configuration space is divided into parts consisting of physically legal and illegal configurations, the so called *free space* and *blocked space*. The physically legal configurations of a kinematic scene are those configurations, in which the objects are placed in the physical space without overlapping.
If in our example the teeth of the gears affect such that a rotation of one gear forces a rotation of the other, then the free space consists of several strips around the torus. If we are only interested in one connected component, we can regard the free space topologically as a circle line, corresponding to a simultaneous rotation of both gears.

In this work we investigate the topological structure of the free spaces of kinematic scenes.
In order to demonstrate that understanding and characterizing properties of free spaces is both a relevant and interesting subject of research, we will give a motivation of our topological investigations from the viewpoint of qualitative reasoning in section 2.
In section 3, we will give a brief mathematical introduction to topological invariants of arbitrary spaces and discuss their computability with respect to applications in robotics. Especially, we will introduce the fundamental group as a concept allowing to give an overview of all possible, essentially different motions in a given kinematic scene. Based on a representation of the free space as a topological cell complex, i.e. on Collin's cell decomposition, a relatively simple procedure allows to compute a finite representation of the fundamental group by generators and relations.
In section 4, we will investigate the structure and the complexity of the fundamental group by studying some simple classes of scenes. This study will reveal, that the minimal number of generators needed to represent the fundamental group of a given scene may grow at a more than exponential rate with respect to the number of movable objects.
After this, we will turn to the question if the introduced invariants can be used to compare or to classify mechanisms. The answer will be provided by a construction which shows that even simple mechanisms have very arbitrary free spaces whose structure according to the invariants in consideration cannot be interpreted in the sense of Turing-Decidability. In section 5, we will introduce axis-parallel, cubical complexes as a computationally handy model of general topological spaces, and in section 6, we will show, that any axis-parallel, cubical complex may occur as free space of a kinematic scene. This adds a topological counterpart to the known intractability results in the robotics and motion planning literature.
In section 7, we discuss the undecidability results that follow from our construc-

tion and how they may be relativized.

2 Motivation

When reasoning about the physical world, people usually use qualitative arguments. If we want e.g. to answer the question what happens, if we throw a ball upwards, we may use the following argumentation: after throwing the ball, it has positive speed, because of the earth gravitation there is a negative acceleration, thus the speed will get zero and turn negative, and the ball will fall down to earth again. For this argumentation, the exact relationships between or the quantitative values of the parameters are not needed.
Qualitative reasoning is a growing field in Artificial Intelligence that is concerned with a description of physical processes using qualitative terms, aiming to provide the next generation of expert systems with more "human skills".
Pursued goals are e.g. to explain or to predict the behaviour of a given physical system, or even to give an envisionment of all possible behaviours. Existing programs try to achieve this by detecting qualitatively different states and by constructing a diagram which represents possible state transitions ([1]). In our small example, we would e.g. want to distinguish between a state when the ball is moving upwards, when it has reached its highest point and when it falls down again.
Qualitative kinematics addresses the problem of reasoning about mechanical systems. Direct goals are e.g. to help engineers to understand or to evaluate mechanisms, with applications in analyzing CAD layouts. This general task can not be achieved by classical numerical simulation techniques: the behaviour of a complex mechanism may depend on several parameters, but simulation only traces a path in the free space starting at a given initial configuration.
The generally used method to generate a representation of all possible behaviours is similar to the approach stated above: one tries to characterize the working of mechanisms by identifying operating modes and mode transitions. To get a distinction of the configurations in qualitative states, the free space is decomposed into regions satisfying adequate criteria such as convexity or monotonicity with respect to specific parameters.
The difficulty of the decomposition technique is the high computational complexity: the complexity of the free space and especially the number of necessary regions to decompose it may grow exponentially in its dimension. Thus the existing efforts based on this technique use pragmatic approaches, e.g. exploiting the fact that most mechanisms are built modularly of simple subassemblies ([2]) or taking into account acting forces ([3]). However, these methods can only deal with special classes of "feasible" mechanisms (shown to be fairly large in [2]), and in order to get a manageable output, further abstraction remains necessary to reduce complexity.
In [4], Joskowicz describes a technique for simplification and abstraction of kinematic behaviour, which is based on a set of simplification rules deduced from practical observations. To deal with imprecise metric information and to de-

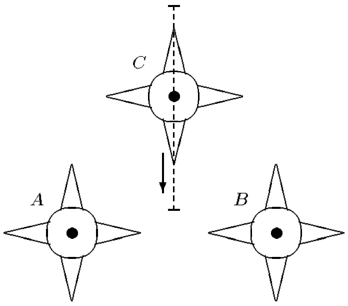

Figure 1: A simple coupling mechanism: by moving the gearwheel C between A and B, the rotary motions of A and B are coupled.

velop a purely qualitative kinematics, Faltings, Baechler and Primus present in [5] a reasoning method based on the "kinematic topology" of mechanisms, denoting connectivity properties of the free space or resp. of a corresponding graph representation.
While these approaches are driven by practical needs, we investigate topological invariants from a theoretical point of view, as a mathematical method providing more subtle abstraction than pure connectivity information, but naturally being still too weak to be a feasible reasoning method without the support of geometric argumentation.

To illustrate the use of topological invariants, we study as a small example two gearwheels A and B in the plane, which can be coupled by moving a third one C between them, this motion being restricted to linear translation (cf. figure 1). This situation can e.g. be found as a simple coupling mechanism in a tape recorder, one gear attached to the motor by a kinematic chain, and the other driving the tape (in old tape recorders such couplings are established mechanically by pressing the operating mode buttons $REW/FF/PLAY$).

The topological structure of the free space for fixed positions of the gear C gives us a distinction of these positions in three intervals:

If the teeth cannot collide, the free space is a 3–dimensional torus. If the motions of the gears A,C and B are fully coupled, the free space corresponds to a circle line. Between both extremes there is a region of interplay where the free space consists of a torus with holes corresponding to collisions of the teeth tips.

A decomposition of the free space in e.g. convex regions would have a complexity $O(n^3)$, if each gearwheel has n teeth. This would even be further increased when taking into account kinematic chains attached to the gears: each tooth in such a chain generally affects the decomposition. Thus the resulting description by a transition diagram may get too detailed for practical problems such as classification.

But we observe that the change of the topological structure of the free space describes the function of the coupling mechanism. Topological considerations thus could be useful when dealing with the question of mechanism understanding, e.g. as a tool to distinguish qualitatively between parameter values or to perform further abstraction in a region decomposition.

While connectivity of the free space can provide only a weak tool to reason about mechanisms, taking into account all knowledge about the geometric shape may be too restrictive. Topology as a mathematical discipline can be looked upon as an effort to perform qualitative reasoning about geometric structures, providing properties of the free space lying between both extremes. Therefore it is natural to ask if it is feasible to characterize or even to *compare* mechanisms according to such properties. Is it possible to construct *normal forms* to arbitrary mechanisms with respect to topological invariants of their free space ?

3 Topological Invariants

To investigate topological invariants of the free spaces of given kinematic scenes with respect to computability, we first have to specify the notion of kinematic scenes.

Definition 1 *A* kinematic scene *is a finite set of objects, each object being a semi-algebraic, connected subset of* $\mathbf{R}^d$ $(d = 2, 3)$*, described by a boolean combination of inequalities of the form* $p(x_1, \ldots, x_d) > 0$ *or* $p(x_1, \ldots, x_d) \geq 0$*, with* p *a multivariate polynomial with integer coefficients.*
Each object is specified either as an obstacle, having no freedom of motion, or as a movable object, the possible motions corresponding to the group of rigid transformations of $\mathbf{R}^d$*.*
The free space *is the space consisting of all those configurations (i.e. transformations), in which the intersection of any pair of the transformed objects, considered as point sets in* $\mathbf{R}^d$*, is empty.*

One should mind that, for the sake of simplification, objects may have "open" and "closed" sides corresponding to the two kinds of inequalities "$>$" and "$\geq$". We will make implicit use of this feature in our constructions, but remark, that the results in this paper can also be obtained for other models, in which e.g. all objects are required to be closed and contacts are allowed.

Under the general suppositions of definition 1 the configuration space of a given scene with m movable objects equals $(\mathbf{R}^d \times SO(d))^m$, $SO(d)$ denoting the orthogonal group of $d \times d$-matrices, and hence can be embedded in $\mathbf{R}^{m \cdot (d+d^2)}$ as a semi-algebraic subset. The free space can be described by a Tarski formula, and thus be considered as a semi-algebraic subset of some euclidian space $\mathbf{R}^n$ ([7]). Schwartz and Sharir have shown that it is possible to compute the *homology groups* of a compact free space by using Collins' algorithm for "cylindrical algebraic decomposition" to produce a decomposition of this space as a regular cell complex with semi-algebraic cells ([6],[7]).

The homology groups, especially the Betti numbers, describe the geometrical complexity of the free space: figuratively speaking, the k-th Betti number is the number of k–dimensional holes. But this still yields no grave abstraction from geometric shape: if we want to compare mechanisms, we are e.g. interested in topological properties that allow to transform configurations and motions of one mechanism to those of another one and vice versa. Thus it is natural to ask for the two most common characterizations of topological spaces, namely *homeomorphy* and *homotopy equivalence*, for which we will give intuitive definitions now.

As we are only interested in subspaces of $\mathbf{R}^n$ we do not have to take care of how the topology may be defined: the significant topologies, i.e. those based on any norm, are equivalent and so we can speak of continuity in the common sense. Two subspaces of $\mathbf{R}^n$ are called homeomorphic iff there is a bijection f between them such that both f and f^{-1} are continuous. Examples are scalings, rotations and affine transformations. Homeomorphisms between the free spaces of mechanisms allow to map motions, i.e. continuous paths in the free space, bijectively between mechanisms and thus e.g. to translate motion planning problems.
The notion of homotopy equivalence provides us with a weaker subdivision of topological spaces into classes: two subspaces of $\mathbf{R}^n$ are homotopy equivalent iff they can be transformed to each other by a process of "blowing up" and "shrinking" (more formally: iff there exists another space which contains both as deformation retracts). This allows us to compare paths in the free spaces of mechanisms up to continuous deformations. As in practical applications production tolerances generally have to be considered, this seems to be a more adequate criterion for classifying mechanisms than homeomorphy.

A useful invariant of homotopy equivalence is the *fundamental group*. We consider paths in a subspace $X \subseteq \mathbf{R}^n$ to be equivalent iff they are homotopic, i.e. iff they can be transferred to each other by a continuous deformation process in X. The fundamental group $\pi_1(X, x)$ consists of the homotopy classes of all loops at a fixed base point x. The group multiplication is induced by the connection of loops. If X is connected (i.e. path-connected), the base point x is redundant. By the fundamental group of a kinematic scene, we denote the fundamental group of the connected component of its free space we are interested in.
If we ask for an overview of the possible motions of a complex mechanism, a representation of the fundamental group by generators and relations can be helpful: the generators give an overview of the essential different cyclic motions, the relations describe the degree of interaction between the parts. This may be illustrated by two gearwheels, each having exactly one tooth: if the teeth of both gearwheels can collide such that the free space corresponds to a torus with one hole in its surface, then the fundamental group of this scene is the free group with two generators a, b, each one corresponding to a rotation of a gear. If no collision is possible, the fundamental group turns to the free abelian group with two generators, i.e. a group with generators a, b and relation $ab = ba$ stating that a and b are "independent" such that their order may be exchanged.

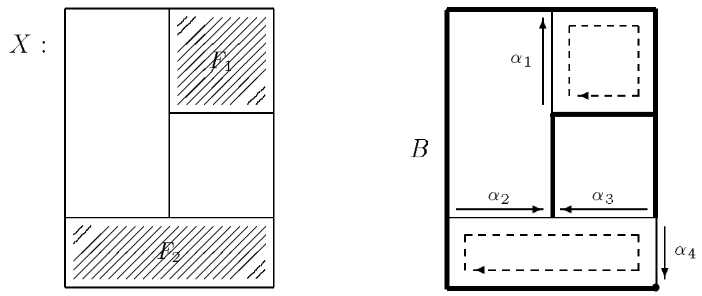

Figure 2: An example for the computation of the fundamental group of a given space X (here consisting of lines and two faces): $\pi_1(X) =< \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid \alpha_1 = 1\ ,\ \alpha_2\alpha_3^{-1}\alpha_4 = 1\ > \approx <\ \beta_1, \beta_2 \mid -\ >$.

By Collins' algorithm, it is possible to compute a cell decomposition of the free space of a given kinematic scene and the neighbourhood relationships between the cells. In our model, the free space may have open sides and be unbounded, and thus the computed decomposition may be no cell complex. But it is easy to render this decomposition to get a topological cell complex, describing a compact space which is homotopy equivalent to the free space ([8]). The representation of the free space as a cell complex X is the basis to compute a finite representation of its fundamental group. The algorithm, which is often denoted as Poincaré's procedure, uses the fact, that the fundamental group is fully determined by the edges and faces of the complex. Figure 2 shows the computation of the fundamental group for a 2–dimensional example space, the general algorithm can be sketched as follows:

1. Consider the 1–dimensional skeleton X^1 of the cell complex X to be a graph and compute a spanning tree B of X^1.
2. Direct the edges of $X^1 \setminus B$ arbitrarily and mark them with symbols $a_1, \ldots, a_k$, the *generators* of the fundamental group.
3. For each face F in the 2–dimensional skeleton X^2 choose a closed path in X^1 around this face. If this path enters edges marked with $a_{i_1}, \ldots, a_{i_l}$ in this order let $\gamma_F := a_{i_1}^{\varepsilon_1} \ldots a_{i_l}^{\varepsilon_l}$; $\varepsilon_j = \pm 1$ denoting the direction in which the path runs through edge a_{i_j} according to its orientation. The relations $\gamma_F = 1$ form the *relations* of the fundamental group.

Theorem 1 *It is possible to compute a finite representation of the fundamental group of a given kinematic scene by generators and relations.*

The running time of the entire algorithm is determined by the running time of Collin's algorithm, which is doubly exponential in the dimension of the free

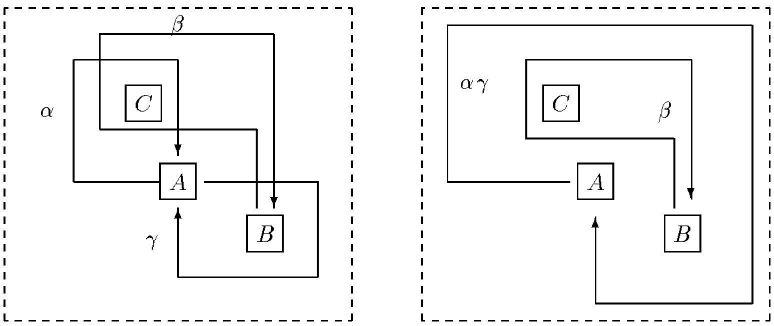

Figure 3: The generators of the fundamental group of a scene with three movable boxes can be interpreted as the motions α, β and γ of single boxes. As the motions $\alpha\gamma$ and β do not "disturb" each other, we get the relation $(\alpha\gamma)\beta = \beta(\alpha\gamma)$.

space. In the next section we will thus focus on the possible structure and complexity of the fundamental group.

4 Fundamental Groups of Simple Scenes

In this section we consider planar scenes consisting of rectangular objects with only translational freedom of motion. We first investigate the structure of the fundamental group, if the scene consists of sliding boxes in the unrestricted plane (cf. figure 3).

If the scene consists of just one box, then the free space is the plane $\mathbf{R}^2$, homotopy equivalent to a point, and the fundamental group is the trivial group.

If the scene consists of two boxes, then the free space is homotopy equivalent to the 4–dimensional space $\mathbf{R}^4$, from which a plane has been removed. Using simple projections, this space can be retracted to a 3–dimensional space without a line and this again to a plane without a point. The situation thus corresponds to the situation when the scene consists of one movable box and one obstacle box. The fundamental group is the free group with one generator, corresponding to a circular motion of one box around the other.

If we consider three movable boxes A,B and C, the situation becomes difficult: again it is easy to see, that this corresponds to the case that one of the boxes is fixed. In both cases the free space is homotopy equivalent to the 4–dimensional space, from which three planes intersecting in one point have been removed. By a complicated calculation using the theorem of Seifert and van Kampen, it can be shown ([8]), that the fundamental group of this space has the representation

$$< \alpha,\beta,\gamma \mid (\alpha\gamma)\beta = \beta(\alpha\gamma),\ (\gamma\beta)\alpha = \alpha(\gamma\beta),\ (\beta\alpha)\gamma = \gamma(\beta\alpha) > .$$

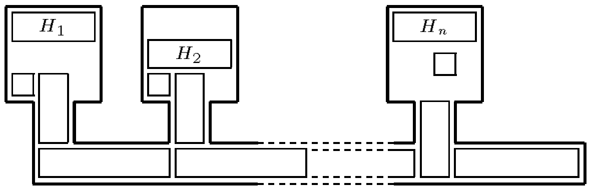

Figure 4: A scene showing that by successively fixing objects (the rectangles $H_1, \ldots, H_n$), the number of generators of the fundamental group may be more than doubled in each step.

The generators can be interpreted as circular motions of A around B, of A around C and of B around C, the relations can be interpreted as "independence" of motions (cf. figure 3).

For the case of n movable boxes, we get $\frac{1}{2}n(n-1) = O(n^2)$ generators and similar relations describing "commutability".

In the following we will investigate, how the fundamental group and especially the number of its generators changes if we fix some of the movable boxes in the plane, i.e. if we consider scenes with obstacles. A process of replacing objects by obstacles corresponds to transitions of the free space to lower–dimensional, linear subspaces. The previous example with three boxes demonstrates, that the number of the generators may remain unchanged or be reduced by fixing boxes. But is the contrary also possible, or is $O(n^2)$ an upper bound ?
The scene in figure 4 is an example which shows, that the number of generators may even be more than doubled in each step of such a process. If all the boxes inside the frame are free to move, the free space is simply connected, i.e. all motions can be transformed to each other by a continuous deformation. If we fix k of the objects $H_1, \ldots, H_n$ in their lower position, then the resulting free space is homotopy equivalent to the 1–dimensional skeleton of a k–dimensional cube. The fundamental group of a connected graph with α_0 nodes and α_1 edges is a free group with $\alpha_1 - \alpha_0 + 1$ generators. For $0 < k \leq n$, we thus get $k2^{k-1} - 2^k + 1 = (k-2)2^{k-1} + 1$ generators, which is more than exponential in the number of the movable objects.

The largest number of generators we could detect occurs in the planar "game" consisting of $n^2 - 1$ sliding boxes of size 1×1 arranged in a box of size $n \times n$ (cf. figure 5).
To examine this scene, we define the following classes of configurations: a connected set of configurations, in which all columns (rows) but one are filled with n boxes, is called a column-(row-)cluster; a discrete configuration, in which there are one column and one row filled with only $n-1$ boxes, is called a permutation-configuration. In the free space, each cluster is the convex hull of n permutation-configurations, and each column-(row-)cluster is neighboured to

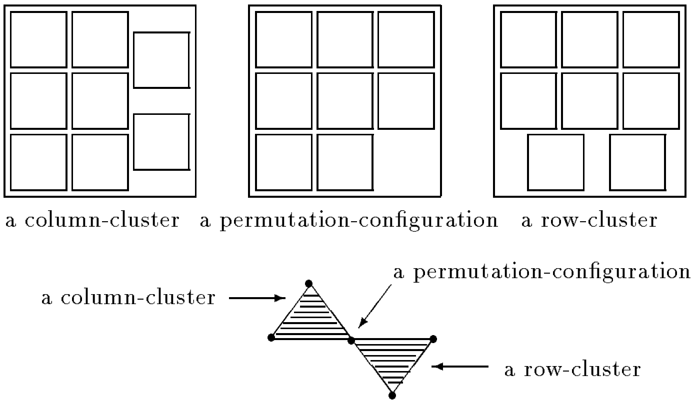

Figure 5: Special classes of configurations in the game of movable boxes and the structure of the free space.

n row-(column-)clusters. The number of permutation-configurations, which can be reached from a given initial configuration, is $n^2!/2$, and the total number of reachable clusters can thus be calculated to be $n^2!/n$.
The free space can be retracted to a graph, whose nodes correspond to the permutation-configurations, and whose edges correspond to the clusters. By Euler's formula, we get for the number of generators of the fundamental group

$$\alpha_1 - \alpha_0 + 1 = \frac{n^2!}{2} - \frac{n^2!}{n} + 1 > (n^2 - 1)!.$$

Theorem 2 *The minimal number of generators needed to represent the fundamental group of a given kinematic scene may grow at the rate faculty of the number of the movable objects (respectively the dimension of the free space).*

As the big number of generators in this game corresponds to our intuition, that motion planning in this scene is more complicated than in the case of boxes moving in the unrestricted plane, it is an interesting question in how far the fundamental group can be used as a good measurement for the complexity of planning problems. Unfortunately, we do not know much about the groups that occur between both investigated extremes, which may be described as the cases of "unrestricted" and "fully restricted" freedom of motion. Interesting questions are e.g. if there are algebraic characterizations for these groups, or if the word problem is decidable. We will see in the next sections, that the latter question has a negative answer, if we consider non-convex objects.

In the previous section we have seen, that it is possible to compute a representation of the fundamental group of an arbitrary kinematic scene, even though with high computational complexity. But we still do not know, if we also can

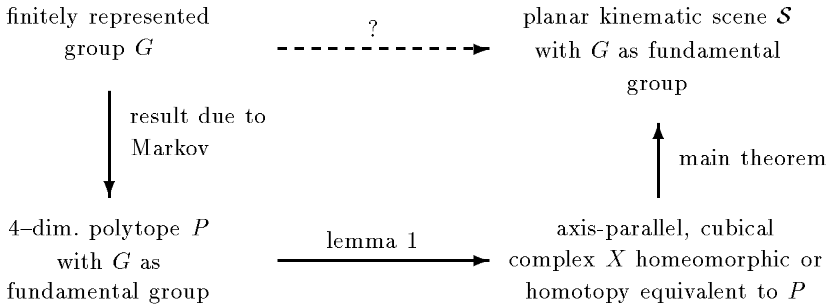

Figure 6: The way from groups to scenes.

interpret this representation, i.e. if we are e.g. able to detect algebraic properties or to use the representations to compare mechanisms. These problems are closely related to the problems of homeomorphy and homotopy equivalence, and we will attack them in the next sections.

5 Classification Problems for Free Spaces

Starting from the fact that there is no algorithmic solution to the word problem for finite group representations, a huge variety of group theoretic problems have been proved to be undecidable. By a construction showing that each finite group representation can occur as fundamental group of a 4–dimensional polytope, Markov could reduce the problem of homeomorphy to the isomorphism problem of groups ([9]). With similar constructions, Boone, Haken and Poénaru could show the undecidability of further topological classification problems such as the problem of homotopy equivalence ([10]).

So if we want to classify kinematic scenes according to the stated topological invariants of their free space, we first have to answer the question, how "arbitrary" the free spaces of the considered scenes can be:

Is it possible to construct for any given 4–dimensional polytope a "real" mechanism whose free space is homeomorphic / homotopy equivalent to the given polytope ?

(Remark that this also would imply that for any given finite group representation one can construct a mechanism which has the represented group as its fundamental group.)

Our main theorem answers this question with yes - thus giving a negative answer to the classification problem.

In order to prove this result, we take a roundabout way via simple subsets of the euclidian space built up by axis-parallel boxes, i.e. boxes for which each boundary edge is parallel to one of the coordinate axes (cf. diagram 6). Furthermore, we require the considered sets to be compact and to be embedded in the unit cube $[0,1]^n$. Analogous to simplicial complexes, we define axis-parallel, cubical complexes:

Definition 2 *An* axis-parallel, cubical complex $X \subseteq \mathrm{R}^n$ *is a set of disjoint, axis-parallel, open boxes of the form* $B = I_1 \times \ldots \times I_n$, I_j *being an open interval or a point interval* $\forall\ 1 \leq j \leq n$, *such that the sides of each box are again contained in* X *(if* $I_j = (a,b)$, *then* $I_1 \times \ldots \times I'_j \times \ldots \times I_n$ *with* $I'_j = \{a\}$ *or* $I'_j = \{b\}$ *is a side of* B*).*
For simplicity, we also denote with X *the union of all boxes in* X.

The following lemma states that the confinement to axis-parallel, cubical complexes is no essential restriction with respect to the topological invariants we consider:

Lemma 1 *It is possible to construct for each finite polytope* $P \subseteq [0,1]^n$ *(i.e. consisting of finitely many simplices) a finite, axis-parallel, cubical complex* $X \subseteq [0,1]^n$ *such that* X *is homotopy equivalent to* P*. In addition, if* P *is an* n*–dimensional manifold (i.e. each point has a neighbourhood homeomorphic to* R^n*), then one can construct* X *as an* n*-manifold homeomorphic to* P*.*

Sketch of proof:
First we have to render P to an n–dimensional manifold homotopy equivalent to P. This can be achieved by triangulating the unit cube $[0,1]^n$, such that the triangulation contains P ; by constructing the second normal subdivision of this triangulation and then taking only those (closed) simplices of the subdivision, which share at least one point with P.
By repeatedly cutting off small enough axis-parallel boxes from the boundary of the resulting n–dimensional polytope $P' \subseteq [0,1]^n$, it is easy to obtain a finite, axis-parallel, cubical complex $X \subseteq [0,1]^n$ homeomorphic to P' (inductively cut off first the points, then the edges, then the 2–dim. faces etc.). $\Box$

In order to carry over the known undecidability results from group theory and topology to fundamental groups and free spaces of kinematic scenes it suffices to prove the following main theorem:

Theorem 3 *For each finite, axis-parallel, cubical complex* $X \subseteq [0,1]^n$ *it is possible to construct a planar kinematic scene consisting of polygonal objects with purely translational freedom of motion, such that the free space of this scene is homotopy equivalent to* X*. Furthermore, if* X *is an* n*–dimensional manifold, the closure of the free space of this scene is homeomorphic to* X*.*

This theorem completes diagram 6: Markov's proof of the undecidability of the homeomorphism problem works with a 4–dimensional polytope in a 4–dimensional space and thus can be carried over to the closure of free spaces; concerning homotopy equivalence and fundamental groups we do not have to take care about the dimensions, i.e. the polytope may be embedded in a more than 4–dimensional space.

The reason, why we confine our investigations to axis-parallel, cubical complexes is because they allow a recursive decomposition into lower–dimensional parts (cf. figure 7).

Lemma 2 *If X is a finite, axis-parallel, cubical complex in $[0,1]^n$, $n \geq 2$, then there exists a decomposition*

$$X = X^{[1]} \times I_1 \dot{\cup} \ldots \dot{\cup} X^{[l]} \times I_l$$

such that $(I_1, \ldots, I_l)$ is a decomposition of $[0,1]$ into intervals and $\forall\ 1 \leq j \leq l$, $X^{[j]}$ is an axis-parallel, cubical complex in $[0,1]^{n-1}$.

By considering the intervals $I_1, \ldots, I_l$ as nodes of a tree, it is possible to describe the n–dimensional complex by a planar structure:

Definition 3 *Let $X \subseteq [0,1]^n$ be an axis-parallel, cubical complex. If $n = 1$, let $(I_1, \ldots, I_l)$ be a decomposition of $[0,1]$ with $I_j \cap X = I_j$ or $I_j \cap X = \emptyset$, $1 \leq j \leq l$; if $n > 1$, let $X = X^{[1]} \times I_1 \dot{\cup} \ldots \dot{\cup} X^{[l]} \times I_l$ according to lemma 2.*
A decomposition tree *$Z(X)$ is a partial function from the set of all finite tupels $\mathbf{N}^*$ (paths) to the set of all subintervals of $[0,1]$, recursively defined by*

$$\begin{aligned} Z(X)[j] &:= I_j \ ,\ 1 \leq j \leq l; \\ Z(X)[j, i_1, \ldots, i_r] &:= Z(X^{[j]})[i_1, \ldots, i_r] \ ,\ 1 \leq j \leq l \ ,\ r < n. \end{aligned}$$

Any path $[i_1, \ldots, i_r]$ of length $r \leq n$ in the tree corresponds to a subbox $[0,1]^{n-r} \times Z(X)[i_1, \ldots, i_r] \times \ldots \times Z(X)[i_1, i_2] \times Z(X)[i_1] \subseteq [0,1]^n$. The intersection of this subbox with X does not "depend" on the last r variables, i.e. the projection of the intersection to the first $n-r$ variables does not change when $(x_{n-r+1}, \ldots, x_n)$ vary over $Z(X)[i_1, \ldots, i_r] \times \ldots \times Z(X)[i_1]$. This projection equals $X^{[i_1, \ldots, i_r]} := (\ldots((X^{[i_1]})^{[i_2]})\ldots)^{[i_r]}$, and is decomposed in the $r+1$-th recursion step according to the $(n-r)$–th variable.
The subboxes $Z(X)[i_1, \ldots, i_n] \times \ldots \times Z(X)[i_1]$ corresponding to all paths from the root to the leaves of the decomposition tree constitute a decomposition of the unit cube into boxes belonging either to X or to $[0,1]^n \setminus X$. Accordingly, we mark the leaves with "$\in X$" and "$\notin X$".
We illustrate the decomposition of axis-parallel, cubical complexes by the example of the surface of the cube $X = \partial[0,1]^3$ (cf. figure 7). This example will be continued in the next section to illustrate the construction of scenes for given free spaces.
If we scan according to the last variable x_3 across the unit cube, then the intersection of the scan plane with X changes at only finitely many points, the endpoints of the decomposition intervals. The intersection consists of the front side for $x_3 \in \{0\}$, of the "frame" for $x_3 \in (0,1)$ and of the back side for $x_3 \in \{1\}$. The decomposition can be continued recursively according to variable x_2. The complete decomposition tree is shown in figure 8.

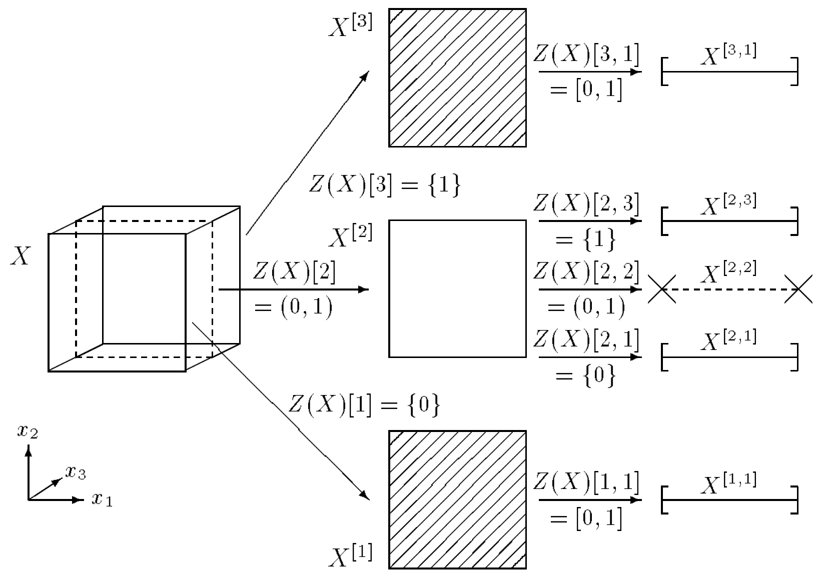

Figure 7: A recursive decomposition of $X = \partial[0,1]^3$.

6 Scenes with Arbitrary Free Spaces

In this section we present a recursive construction method for generating scenes for arbitrary given, finite, axis-parallel, cubical complexes, such that their free space fulfills the requirements of theorem 3.

In the following, we use "n–dimensional space" abbreviating for spaces which are embedded in the n–dimensional euclidian space. Recursively, we will build the scene for an n–dimensional space X by using the scenes for the $(n-1)$–dimensional spaces $X^{[1]}, \ldots, X^{[l]}$. Thus we begin with the construction of scenes for 1–dimensional complexes, i.e. unions of closed intervals.

To construct scenes with a 1–dimensional free space like $X^{[2,1]} = [0,1]$ or $X^{[2,2]} = \{0\} \cup \{1\}$, we use a bar B, which can be moved in the horizontal direction and which is formed such that the area in which the bar may slide can be restricted to the desired intervals by the help of an obstacle Q. The horizontal position of B corresponds to the value of parameter x_1. In figure 9, scenes for $X^{[2,1]}$ and $X^{[2,2]}$ are shown: the thickness of the obstacle Q can be chosen as less than half the length of the smallest forbidden interval. The names of the objects are indexed by the tuple describing the adequate path in the decomposition tree.

To construct a scene for a 2–dimensional free space like $X^{[2]}$, we want to use the already constructed scenes for the 1–dimensional spaces resulting from the decomposition and "activate" them according to the position of a new sliding object representing the value of parameter x_2. How is it possible to choose between $B^{[2,1]}$, $B^{[2,2]}$ and $B^{[2,3]}$? We first connect these bars to a new bar $B^{[2]}$,

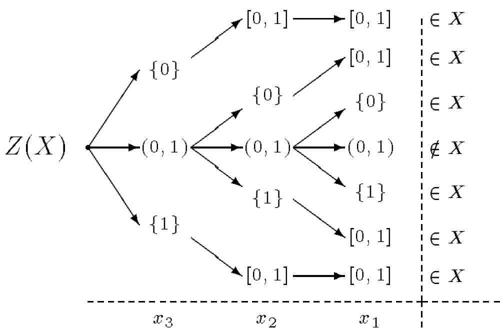

Figure 8: A complete decomposition tree for $X = \partial[0,1]^3$.

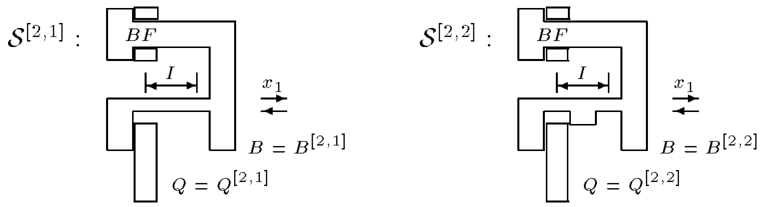

Figure 9: Scenes $\mathcal{S}^{[2,1]}$ / $\mathcal{S}^{[2,2]}$ for the 1–dimensional free spaces $X^{[2,1]}$ / $X^{[2,2]}$.

such that again one parameter x_1 suffices to describe its position (cf. figure 10). By using closed sides as the bottom sides of B and as the top side of Q, we can make it possible that the area of allowed positions of B is enlarged to the entire interval $[0,1]$ by moving Q downwards at an arbitrary small amount. Thus, we substitute the obstacles $Q^{[2,1]}$, $Q^{[2,2]}$ and $Q^{[2,3]}$ by vertically movable "plungers" $QU^{[2,1]}$, $QU^{[2,2]}$ and $QU^{[2,3]}$, whose positions are determined by the position of a "sliding regulator" $SR^{[2]}$.
The task of $SR^{[2]}$ is to ensure, that the object $QU^{[2,i]}$ is in its upmost position, i.e. forces x to be in $X^{[2,i]}$, iff the horizontal position of the regulator corresponds to a value of parameter x_2 belonging to interval $Z(X)[2,i]$. The construction principle is shown in figure 10.

But the shown scene is still faulty, as the sliding regulator $SR^{[2]}$ doesn't fulfill its task completely: the object $QU^{[2,2]}$ should be in its upmost position iff x_2 is in the open interval $(0,1)$, but it really restricts the allowed positions of the bar $B^{[2]}$ for $x_2 \in [0,1]$.

As this problem arises also in the further steps of our inductive technique, the following considerations are anticipatingly formulated for the general case. In our construction, we cannot avoid that a plunger is always in its upmost position during closed intervals of horizontal positions of its regulator. Hence at the intersection $\overline{I_1} \cap \overline{I_2}$ of two intervals $I_1 = Z(X)[i_1,\ldots,i_r]$ and $I_2 = Z(X)[i_1,\ldots,i_r+1]$

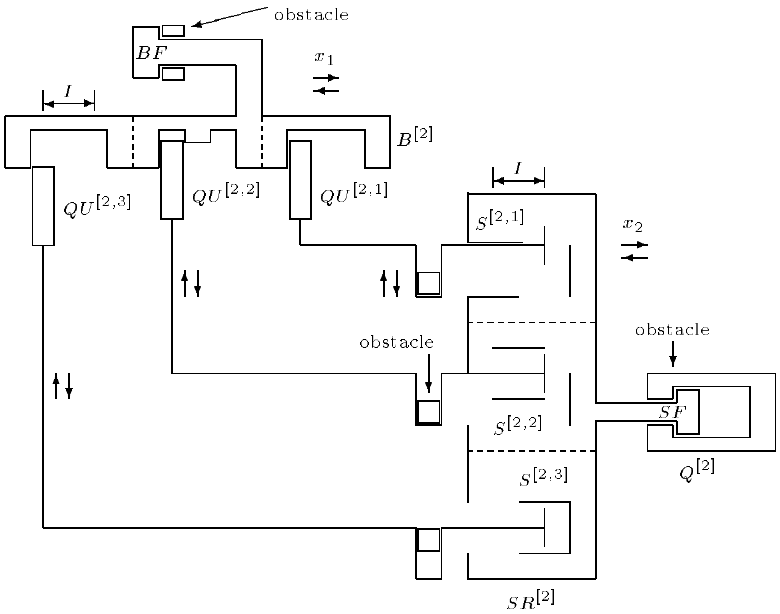

Figure 10: The construction idea for a scene $\mathcal{S}^{[2]}$ with $X^{[2]}$ as free space.

both the plungers of the scene for $X^{[i_1,\ldots,i_r]}$ and the scene for $X^{[i_1,\ldots,i_r+1]}$ are active, thus forcing the free space to be the intersection of both spaces. Therefore, our problem only affects the border of X, i.e. some closed sides of X may get lost. If X is an n–dimensional manifold, then the closure of the free space naturally equals X. But if $X \subseteq [0,1]^n$ contains parts of dimension lower than n (as in our example), the topological structure may be changed.

In this case, if we are only interested in fundamental groups or homotopy type, it is possible to "cheat" by realizing as free space not X itself but an enlarged version of X.

We start by computing a decomposition tree of X that contains only open intervals or point intervals for the second up to the n-th coordinate. As the problematic cases can only occur at the point intervals, we replace all of them by open intervals:

$$\begin{array}{cccccccc} & \ldots & \{a_1\} & , & (a_1,a_2) & , & \{a_2\} & \ldots \\ \rightarrow & \ldots & (a_1-\varepsilon, a_1+\varepsilon) & , & [a_1+\varepsilon, a_2-\varepsilon] & , & (a_2-\varepsilon, a_2+\varepsilon) & \ldots \end{array}$$

If we choose ε less than half the length of the smallest open interval in $Z(X)$, the resulting tree describes a space Y which is just a blown up version of X and

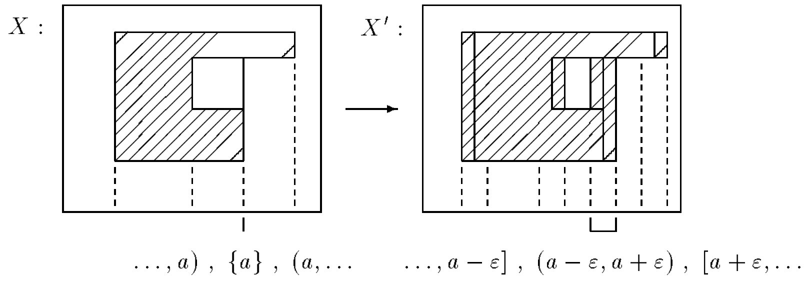

Figure 11: An example for the process of blowing up a space X according to one coordinate.

especially homotopy equivalent to X (cf. figure 11):

$$X'_{(i)} := \{\ x + (0, \ldots, \overrightarrow{0}, \overset{i}{\delta}, \overleftarrow{0}, \ldots, 0)\ ;\ x \in X'\ ,\ -\varepsilon < \delta < \varepsilon\ \} \cap [0,1]^n,$$
$$Y := (\ldots((X_{(n)})_{(n-1)})\ldots)_{(2)}.$$

If we apply our "faulty" construction principle to $Z(Y)$ instead of $Z(X)$, such that for each decomposition interval $Z(Y)[i_1, \ldots, i_r]$, $r < n$, the corresponding plunger $QU^{[i_1,\ldots,i_r]}$ is in its upmost position for $x_{n-r+1} \in \overline{Z(Y)[i_1, \ldots, i_r]}$, then the free space of the resulting scene is just the space Y described by $Z(Y)$ and defined above:
Let $I_1 = Z(Y)[i_1, \ldots, i_r] = (a, b)$ and $I_2 = Z(Y)[i_1, \ldots, i_r + 1] = [b, c]$. Then $Z(X)[i_1, \ldots, i_r]$ has been a point interval and thus $X^{[i_1,\ldots,i_r+1]} \subseteq X^{[i_1,\ldots,i_r]}$ (X is closed !). Hence, also $Y^{[i_1,\ldots,i_r+1]} \subseteq Y^{[i_1,\ldots,i_r]}$ and the allowed parameters $(x_1, \ldots, x_n)$ in the subbox $[0,1]^{n-r} \times Z(Y)[i_1, \ldots, i_r] \times \ldots \times Z(Y)[i_1]$ at $x_{n-r+1} = b$ are just $Y^{[i_1,\ldots,i_r]} \cap Y^{[i_1,\ldots,i_r+1]} = Y^{[i_1,\ldots,i_r+1]}$. As b belongs to interval $I_2 = Z(Y)[i_1, \ldots, i_r + 1]$, the free space of our construction will be exactly the space Y described by $Z(Y)$.
Applied to our example, we get the situation shown in figure 12. Remark, that some sides of the blown up space Y may be open.

In the last construction step of our example we will build a scene $\mathcal{S}$ for the entire space Y (respectively X) by using the scenes $\mathcal{S}^{[1]}$, $\mathcal{S}^{[2]}$ and $\mathcal{S}^{[3]}$ for $Y^{[1]}$, $Y^{[2]}$ and $Y^{[3]}$. The first action is - analogous to the second step - to connect the bars to one bar B. Again we will "activate" the scenes according to the parameter x_3 which is represented by the horizontal position of a new sliding regulator SR. This is achieved by moving the entire scenes (i.e. all regulators and plungers) up and down by SR in such a way that the plungers of scene $\mathcal{S}^{[i]}$ are able to restrict the allowed positions of the bar B iff the position of SR corresponds to a value of x_3 in $\overline{Z(Y)[i]}$. For this purpose we substitute the obstacles $Q^{[i]}$ by vertically movable objects $QU^{[i]}$, whose vertical positions are determined by the position of SR. We fix the vertical position of SR again by an obstacle Q - thus making further recursion possible.

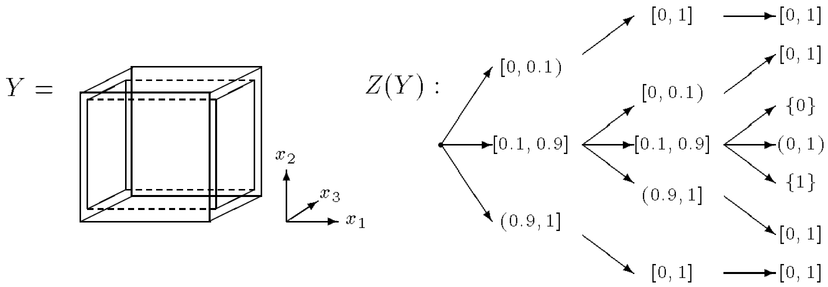

Figure 12: The blown up space Y and its decomposition tree.

By connecting the bars $B^{[1]}$, $B^{[2]}$ and $B^{[3]}$ to the new bar B, we could achieve that only one single parameter x_1 determines the position of all the subbars. In this step, we also have to ensure that only one parameter x_2 determines the horizontal positions of the regulators $SR^{[1]}$, $SR^{[2]}$ and $SR^{[3]}$. As these regulators must be able to move up and down independently, a rigid connection is not possible. Thus we use telescopic connection parts, as shown in figure 13.

It is easy to see that the construction process can be continued to get scenes for spaces of arbitrary dimension, especially of dimension 4. The resulting scenes are similar to the decomposition tree, in fact instead of a recursive construction we could have built the scenes directly starting from this tree. An interval $Z(Y)[i_1,\ldots,i_r]$ corresponds to a regulator $SR^{[i_1,\ldots,i_r]}$ iff $1 \leq r \leq n-2$ and to an elementary subbar $B^{[i_1,\ldots,i_r]}$ iff $r = n-1$. The root at depth zero corresponds to SR. The horizontal position of all regulators at depth r corresponds to parameter x_{n-r}, the horizontal position of the bar B corresponds to x_1. If for $r \leq n-2$ interval $Z(Y)[i_1,\ldots,i_r]$ has λ sons in the decomposition tree, then $SR^{[i_1,\ldots,i_r]}$ consists of λ parts $S^{[i_1,\ldots,i_r,1]},\ldots,S^{[i_1,\ldots,i_r,\lambda]}$ such that the j–th part forces $QU^{[i_1,\ldots,i_r,j]}$ to be in an upper position iff $x_{n-r} \in \overline{Z(Y)[i_1,\ldots,i_r,j]}$ (cf. figure 14).

Remember, that a path in the tree from the root to a leaf corresponds to a subbox which either belongs to Y or to $[0,1]^n \setminus Y$. If the parameters that determine the configuration of the scene belong to the subbox $Z(Y)[i_1,\ldots,i_n] \times \ldots \times Z(Y)[i_1]$, then the plunger $QU^{[i_1,\ldots,i_{n-1}]}$ is in its upmost position and collides with a "tooth" of the bar B iff this subbox belongs to $[0,1]^n \setminus Y$.

The proportions of the objects, especially of the telescopic connection parts, depend on their "depth" in the construction (the vertical movement of the regulators at depth r can be larger than at depth $r-1$).

These considerations complete our construction, which leads to scenes having the properties required in the main theorem.

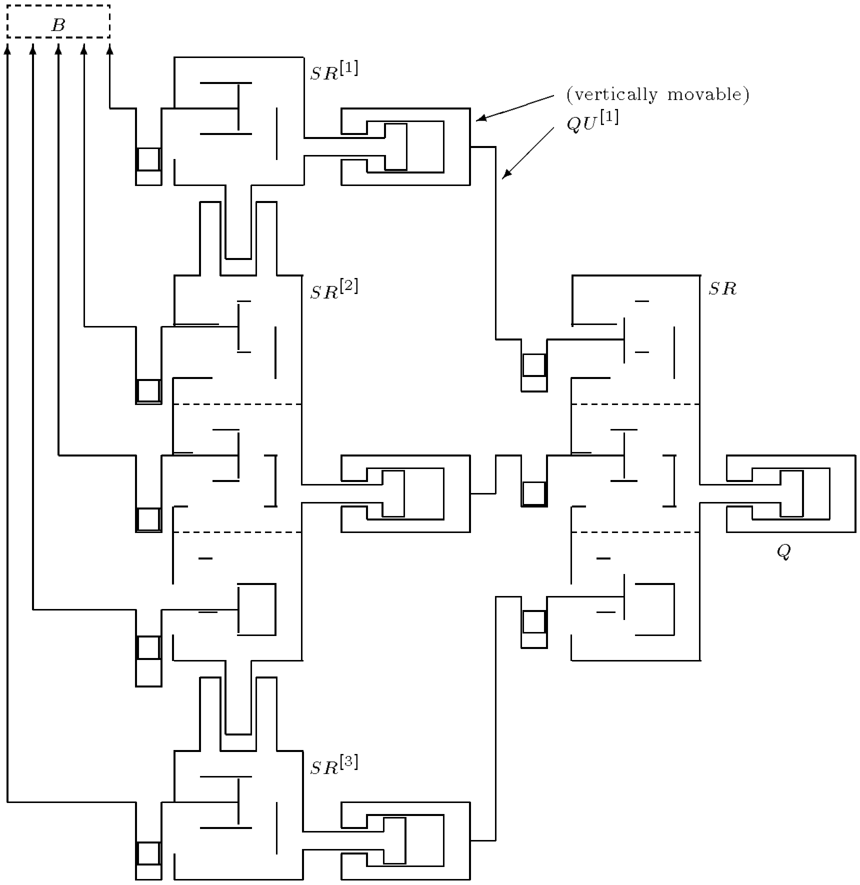

Figure 13: The scene $\mathcal{S}$ with Y as free space: the bar B is indicated by the dashed box, the arrows correspond to plungers.

7 Results

In this section we will give a brief overview of undecidability results that follow from our construction and discuss them.

Of the known undecidability results from group theory ([11],[12]), which all can be carried over to fundamental groups of kinematic scenes, we state the following: It is not decidable, if

- the fundamental groups of two given kinematic scenes are isomorphic;
- the fundamental group of a given kinematic scene has a fixed given "Markov property" such as finiteness, triviality or commutativity;
- the fundamental group is a non-trivial direct/free product.

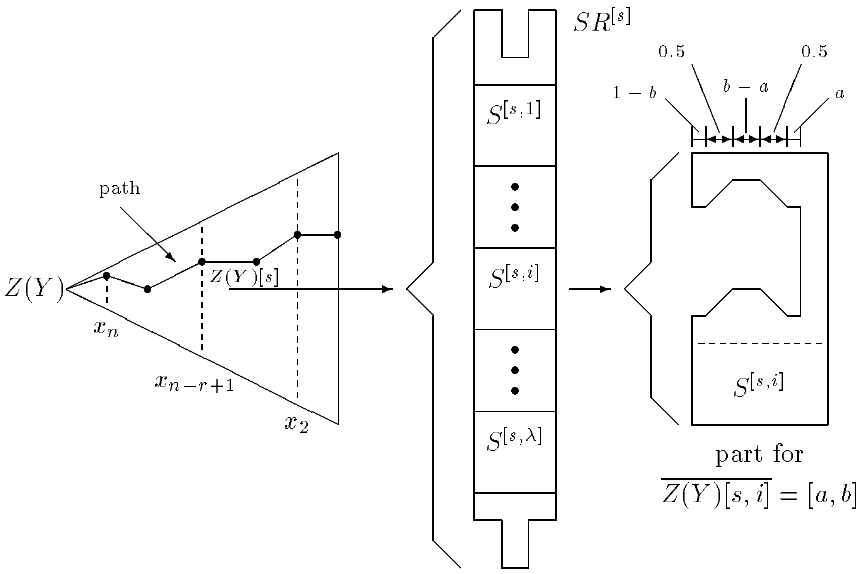

Figure 14: The relation between the decomposition tree $Z(Y)$ and the scene for Y.

Thus the fundamental group does not provide us with a useful tool to get an envision of the "structure" of an arbitrary mechanism.

The classification problem from the motivation - to compare mechanisms according to the topological structure of their free space or to construct adequate normal forms - is undecidable for the most common invariants homeomorphy and homotopy equivalence.

Nevertheless, concerning normal forms we get the interesting result that rotations do not increase the "topological complexity" of kinematic scenes: for each kinematic scene in which there are objects with rotational freedom it is possible to construct a "comparable" planar scene (i.e. according to homeomorphy/homotopy) in which all objects have only translational freedom of motion (to prove this, one has to use Hironaka's result [13] that each semi-algebraic manifold can be triangulated and thus be transformed to a polytope).

A further interesting undecidability result concerns the word problem of fundamental groups, which can be carried over to the homotopy problem of paths in the free space of a given kinematic scene: we even can state that there is a fixed scene such that it is not decidable, if a loop in its free space (i.e. a cyclic motion) is homotopic to the trivial loop staying at the base point. If we want to optimize a given path according to an optimality criterion such that the trivial loop is always more optimal than any other loop, then we cannot find the optimum in the homotopy class of this path because else we could solve the word problem for arbitrary groups. Thus deformation algorithms can never guarantee to find global optima of this kind (e.g. paths of minimal length).

Concluding this section, we should relativize the negative answers to the questions concerning the applicability of "topological reasoning" as a tool to support qualitative reasoning about physical devices.
We remark, that undecidability occurs for free spaces of dimension four or more, whereas most "interactions" in mechanisms are due to less than four interacting parts.
Furthermore, it may be possible to adapt the considered topological invariants to the specific problem of reasoning about motions of objects, such that the substantial problems get decidable: as motions, which are repeated several times, are usually not interesting, it makes e.g. sense to consider fundamental groups, which are factorized by the relations $\alpha^2 = \alpha$ for all generators α.
Finally, the investigation of special cases of kinematic scenes, e.g. the case that all objects are convex, seems to be interesting from both a computer scientific and a mathematical point of view, because some new and interesting classes of groups may be found.

8 Acknowledgements

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